AN ELEMENTARY, ILLUSTRATIVE PROOF OF THE RADO-HORN THEOREM

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Abstract. The Rado-Horn theorem provides necessary and sufficient conditions for when a family of vectors can be partitioned into a fixed number of linearly independent sets. Such partitions exist if and only if every subset of the vectors satisfies the so-called Rado-Horn inequality. In this paper we provide an elementary proof of the Rado-Horn theorem as well as results for the redundant case. Previous proofs give no information about how to actually partition the vectors; we use ideas present in our proof to find subsets of vectors which may be used to construct a kind of “optimal” partition.

1. Introduction

The terminology Rado-Horn theorem was first introduced in [3]. This theorem [12, 15] provides necessary and sufficient conditions for a family of vectors to be partitioned into \( k \) linearly independent sets:

**Theorem 1.1. (Rado-Horn)** Consider a family of non-zero vectors \( \Phi = \{\varphi_i\}_{i=1}^M \) in a vector space. Then the following are equivalent:

(i) The set \( \{1, \ldots, M\} \) can be partitioned into sets \( \{A_j\}_{j=1}^k \) such that \( \{\varphi_i\}_{i \in A_j} \) is a linearly independent set for all \( j = 1, 2, \ldots, k \).

(ii) For any non-empty subset \( J \subseteq \{1, \ldots, M\} \), \( |J| / \dim \text{span}(\{\varphi_i\}_{i \in J}) \leq k \).

The Rado-Horn theorem has found application in several areas including progress on the Feichtinger conjecture [5], a characterization of Sidon sets in \( \Pi_{k=1}^\infty Z_p \) [13, 14], and a notion of redundancy for finite frames [1]. A generalized version of the Rado-Horn theorem has also found use in frame theory where redundancy is at the heart of the subject [2].

Unfortunately, proving the Rado-Horn theorem tends to be very intricate. Pisier, when discussing a characterization of Sidon sets in \( \Pi_{k=1}^\infty Z_p \) states “... d’un lemme d’alébre dû à Rado-Horn dont la démonstration est relativement délicate” [14]. Today there are at least six proofs of the Rado-Horn theorem [4, 5, 10, 11, 12, 15]. The theorem was proved in a more general algebraic setting in [12, 15] and then for matroids in [10]; these proofs are all delicate. Harary and Welsh [11] improved upon the matroid version of the Rado-Horn theorem with a short and elegant proof; however, their argument requires a development of certain deep structures within matroid theory. The Rado-Horn theorem was generalized in [4] to include partitions of a set of vectors with subsets of specified sizes removed, and the

2000 Mathematics Subject Classification. 15A03, 05A17.

Key words and phrases. Rado-Horn; Vector partitions.

The first author is supported by NSF DMS 1008183, NSF ATD 1042701 and AFOSR DGE51: FA9550-11-1-0245.
authors also proved results for the redundant case - the case where a set of vectors cannot be partitioned into \( k \) linearly independent sets. Unfortunately the proofs for these refinements to the theorem are even more delicate than the original. Finally, the Rado-Horn theorem was rediscovered in [5], where the authors give an induction proof which may be considered elementary. This proof has some limitations, however, as it does not clearly generalize nor does it describe the redundant case; it does not reveal the origin of the Rado-Horn inequality.

In this paper, we present an elementary proof which is at the core of the Rado-Horn theorem. With slight modification, these simple arguments prove a generalization of the Rado-Horn theorem and provide results for the redundant case similar to those in [4]. Most appealing, the arguments we present may be thought of visually and provide insight into the specific conditions which give rise to the inequality in the Rado-Horn theorem. These ideas can then be used to construct partitions which contain the fewest possible number of linearly independent sets and which are optimal with regard to certain spanning properties. We will make this clear in the definition of a fundamental partition.

This paper is organized into three sections. The first develops constructions and main arguments used throughout the paper. The second section uses these tools to prove the Rado-Horn theorem, the original and the redundant case. The final section describes which subsets maximize the Rado-Horn inequality and how similar subsets may be used to construct a so-called fundamental partition.

2. Preliminaries

We will always consider \( \Phi = \{ \varphi_i \}_{i=1}^M \) to be a finite family of non-zero vectors in a real or complex vector space. Note this family may contain repeated vectors. Our proof of the Rado-Horn theorem relies on a special partition of this family. In this section we define a so-called fundamental partition and demonstrate several of its remarkable properties.

**Definition 2.1.** Given a family of vectors \( \Phi = \{ \varphi_i \}_{i=1}^M \), let \( \{ A_j \}_{j=1}^k \) be a partition of \( \{ 1, \ldots, M \} \). We call \( \{ \{ \varphi_i \}_{i \in A_j} \}_{j=1}^k \) an **ordered partition** of \( \Phi \) if \( |A_j| \geq |A_{j+1}| \) for all \( j = 1, \ldots, k-1 \).

**Definition 2.2.** Given a family of vectors \( \Phi = \{ \varphi_i \}_{i=1}^M \), let \( \{ P_k \}_{k=1}^m \) be all possible ordered partitions of \( \Phi \) into linearly independent sets. Let \( F_k = \{ \{ \varphi_i \}_{i \in F_{kj}} \}_{j=1}^{r_k} \) so that \( \{ \varphi_i \}_{i \in F_{kj}} \) denotes the \( j \)th set in the \( k \)th partition. Now define

\[
a_1 = \max_{k=1, \ldots, m} |F_{k1}|.
\]

Consider only the partitions \( \{ P_k : |F_{k1}| = a_1 \} \), and define

\[
a_2 = \max_{\{ k : |F_{k1}| = a_1 \}} |F_{k2}|.
\]

We continue so that given \( a_1, \ldots, a_n \),

\[
a_{n+1} = \max_{\{ k : |F_{k1}| = a_1, \ldots, |F_{kn}| = a_n \}} |F_{k(n+1)}|.
\]

When \( \sum_{i=1}^n a_i = M \), any remaining partition is in the set \( \{ P_k : |F_{k1}| = a_1, \ldots, |F_{kn}| = a_n \} \). We call any such ordered partition of \( \Phi \) a **fundamental partition** which we write as \( \{ \{ \varphi_i \}_{i \in F_j} \}_{j=1}^r \). Also, we will use the notation \( \varphi_{(j)} \) when denoting some vector \( \varphi_i \) in \( \{ \varphi_i \}_{i \in F_j} \).
We introduce a fundamental partition as in Definition 2.2 because existence is clear. However, a fundamental partition is a specific example of a basis for a sum of matroids [6, 7]. The following theorem gives a useful alternative definition and is Theorem 1 from [7].

**Theorem 2.3.** Let \( \Phi = \{ \varphi_i \}_{i=1}^M \) be a family of vectors. Then \( \{ \{ \varphi_i \}_{i \in F_j} \}_{j=1}^\ell \) is a fundamental partition if and only if for any other ordered partition \( \{ \{ \varphi_i \}_{i \in A_j} \}_{j=1}^k \) of \( \Phi \) into linearly independent sets,

(i) \( \ell \leq k \)

(ii) \( \sum_{j=1}^n |A_j| \leq \sum_{j=1}^n |F_j|, n = 1, 2, \ldots, \ell. \)

It is helpful to view a fundamental partition as a Young diagram where each square represents a vector, and the rows correspond to the sets \( \{ \varphi_i \}_{i \in F_j} \); see Figure 1. Intuitively, if a Young diagram represents a partition of vectors into linearly independent sets, a fundamental partition is a Young diagram which is as top-heavy as possible.

\[ \begin{array}{c|c|c|c|c|c|c|c} & F_1 & F_2 & F_3 & \ldots & F_\ell \\ \hline F_1 & & & & & & \\ \hline F_2 & & & & & & \\ \hline F_3 & & & & & & \\ \hline \vdots & & & & & & \\ \hline F_{\ell-1} & & & & & & \\ \hline F_\ell & & & & & & \end{array} \]

**Figure 1.** Example of a fundamental partition

Next we will examine spanning properties of a fundamental partition. We will often use the following well known result.

**Proposition 2.4.** Let \( \Phi = \{ \varphi_i \}_{i=1}^M \) be a set of linearly independent vectors. Suppose \( \psi \in \text{span}(\Phi) \) so that \( \psi = \sum_{i=1}^M c_i \varphi_i \). Then for any \( j \in \{1, \ldots, M\} \) such that \( c_j \neq 0 \), \( \Psi_j = (\Phi \setminus \{ \varphi_j \}) \cup \{ \psi \} \) is linearly independent and \( \text{span}(\Psi_j) = \text{span}(\Phi) \).

The following lemma is trivial but does provides some information concerning spanning properties of our partition.

**Lemma 2.5.** Let \( \{ \{ \varphi_i \}_{i \in F_j} \}_{j=1}^\ell \) be a fundamental partition of \( \Phi = \{ \varphi_i \}_{i=1}^M \). Then \( \text{span}(\{ \varphi_i \}_{i \in F_j}) \subseteq \text{span}(\{ \varphi_i \}_{i \in F_r}) \) for \( r \leq j \).

**Proof.** Suppose there existed some \( \varphi_{(j)} \in \{ \varphi_i \}_{i \in F_j} \), such that \( \varphi_{(j)} \notin \text{span}(\{ \varphi_i \}_{i \in F_r}) \). Then

\( \{ \varphi_i \}_{i \in F'_j} = \{ \varphi_i \}_{i \in F_j} \cup \{ \varphi_{(j)} \} \)

is linearly independent with \( |F'_j| > |F_j| \) contradicting our assumption that \( \{ \{ \varphi_i \}_{i \in F_j} \}_{j=1}^\ell \) is a fundamental partition. \( \square \)
This shows that in a fundamental partition, any vector is contained in the spans of the sets before it. Next we show some vectors must be contained in the spans of almost every set.

**Lemma 2.6.** Let \( \{\varphi_i\}_{i \in F_i} \) be a fundamental partition of \( \Phi = \{\varphi_i\}_{i=1}^M \). Pick any \( \varphi_i \in \{\varphi_i\}_{i \in F_i} \) and let \( S_{\ell-1} \subseteq F_{\ell-1} \) be the smallest set such that \( \varphi_i \in \text{span}(\{\varphi_i\}_{i \in S_{\ell-1}}) \). Fix any \( k \leq \ell - 1 \), and let \( S_k \subseteq F_k \) be the smallest set such that \( \text{span}(\{\varphi_i\}_{i \in S_k}) \subseteq \text{span}(\{\varphi_i\}_{i \in S_{\ell-1}}) \). Then \( \text{span}(\{\varphi_i\}_{i \in S_k}) \subseteq \text{span}(\{\varphi_i\}_{i \in F_j}) \), \( j = 1, \ldots, \ell - 1 \).

**Proof.** Clearly the sets \( S_{\ell-1} \) and \( S_k \) exist by Lemma 2.5. We will prove the statement for \( j = \ell - 1 \). The result will then follow for all \( j = 1, \ldots, \ell - 1 \) since \( \text{span}(\{\varphi_i\}_{i \in F_{\ell-1}}) \subseteq \text{span}(\{\varphi_i\}_{i \in F_j}) \) for \( j \leq \ell - 1 \).

We will assume the result fails and get a contradiction. Suppose there exists some \( \varphi_i \in \{\varphi_i\}_{i \in S_k} \) such that \( \varphi_i \notin \text{span}(\{\varphi_i\}_{i \in F_{\ell-1}}) \). By Proposition 2.4, there exists some \( \varphi_i \in \{\varphi_i\}_{i \in S_{\ell-1}} \) such that

\[
\left( \{\varphi_i\}_{i \in S_k} \setminus \{\varphi_i(k)\} \right) \cup \{\varphi_i(1)\}
\]

is linearly independent with the same span as \( \{\varphi_i\}_{i \in S_k} \). Similarly,

\[
\left( \{\varphi_i\}_{i \in S_{\ell-1}} \setminus \{\varphi_i(1)\} \right) \cup \{\varphi_i(1)\}
\]

is linearly independent and has the same span as \( \{\varphi_i\}_{i \in S_{\ell-1}} \). Thus we can partition \( \Phi \setminus \{\varphi_i(k)\} \) into \( \ell \) linearly independent sets, say \( \{\varphi_i\}_{i \in G_j} \) given by

\[
\{\varphi_i\}_{i \in G_j} = \begin{cases} 
\left( \{\varphi_i\}_{i \in F_k} \setminus \{\varphi_i(k)\} \right) \cup \{\varphi_i(1)\} & \text{for } j = k \\
\left( \{\varphi_i\}_{i \in F_{\ell-1}} \setminus \{\varphi_i(1)\} \right) \cup \{\varphi_i(1)\} & \text{for } j = \ell - 1 \\
\{\varphi_i\}_{i \in F_j} \setminus \{\varphi_i(1)\} & \text{for } j = \ell \\
\{\varphi_i\}_{i \in F_j} & \text{for } j \neq k, \ell - 1, \ell.
\end{cases}
\]

Notice \( |G_j| = |F_j| \) and \( \text{span}(\{\varphi_i\}_{i \in G_j}) = \text{span}(\{\varphi_i\}_{i \in F_j}) \) for \( j = 1, \ldots, \ell - 1 \), but then

\[
\{\varphi_i\}_{i \in G_{\ell-1}} \cup \{\varphi_i(1)\}
\]

is also linearly independent with \( |\{\varphi_i\}_{i \in G_{\ell-1}} \cup \{\varphi_i(1)\}| > |\{\varphi_i\}_{i \in F_{\ell-1}}| \). This contradicts the fact that \( \{\varphi_i\}_{i \in F_j} \) was a fundamental partition. \( \square \)

We can extend Lemma 2.6 to obtain a larger set of vectors which must be contained in the spans of each \( \{\varphi_i\}_{i \in F_j}, j = 1, \ldots, \ell - 1 \); this is done by iterating the argument.

**Corollary 2.7.** Let \( \{\varphi_i\}_{i \in F_i} \) be a fundamental partition of \( \Phi = \{\varphi_i\}_{i=1}^M \). Pick any \( \varphi_i \in \{\varphi_i\}_{i \in F_i} \) and for \( j = 1, \ldots, \ell - 1 \), let \( S^{(1)}_j \subseteq F_j \) be the smallest set such that \( \varphi_i \in \text{span}(\{\varphi_i\}_{i \in S^{(1)}_j}) \). Pick a \( k_1 \) so that

\[
|S^{(1)}_{k_1}| = \max_{j=1,\ldots,\ell-1} |S^{(1)}_j|,
\]

and set \( S^{(2)}_{k_1} = S^{(2)}_{k_1} \). Now define \( S^{(2)}_j \subseteq F_j, j = 1, \ldots, \ell - 1 \) as the smallest subset such that \( \text{span}(\{\varphi_i\}_{i \in S^{(2)}_j}) \subseteq \text{span}(\{\varphi_i\}_{i \in S^{(2)}_{k_1}}) \) and choose \( k_2 \) so that

\[
|S^{(2)}_{k_2}| = \max_{j=1,\ldots,\ell-1} |S^{(2)}_j|.
\]
Continue this process so that $S_j^{(n)} \subseteq F_j$, $j = 1, \ldots, \ell - 1$ is the smallest subset such that span($\{\varphi_i\}_{i \in S_{kn}^{(n)}}$) $\subseteq$ span($\{\varphi_i\}_{i \in S_j^{(n)}}$) and choose $k_n$ so that
\[
\left| S_{kn}^{(n)} \right| = \max_{j=1, \ldots, \ell-1} \left| S_j^{(n)} \right|.
\]
Then span($\{\varphi_i\}_{i \in S_{kn}^{(n)}}$) $\subseteq$ span($\{\varphi_i\}_{i \in F_j}$), for $j = 1, \ldots, \ell - 1$.

**Proof.** For $n = 1$, this is Lemma 2.6. Notice this guarantees the sets $S_j^{(2)}$, $j = 1, \ldots, \ell - 1$ are well defined. It now suffices to show span($\{\varphi_i\}_{i \in S_{kn}^{(n)}}$) $\subseteq$ span($\{\varphi_i\}_{i \in F_{\ell-1}}$).

Once again, we proceed by contradiction. Suppose instead there existed some $\varphi_{(k_n)}^{(n)} \in \{\varphi_i\}_{i \in S_{kn}^{(n)}}$ such that $\varphi_{(k_n)}^{(n)} \notin$ span($\{\varphi_i\}_{i \in F_{\ell-1}}$). By Proposition 2.4, there exists some $\varphi_{(k_{m_1})}^{(m_1)} \in \{\varphi_i\}_{i \in S_{kn}^{(m_1)}}$, $m_1 < n$, such that
\[
(\{\varphi_i\}_{i \in S_{kn}^{(n)}} \setminus \{\varphi_{(k_n)}^{(n)}\}) \cup \{\varphi_{(k_{m_1})}^{(m_1)}\}
\]
is linearly independent and has the same span as $\{\varphi_i\}_{i \in S_{kn}^{(n)}}$. Note there may be several such $m_i$ for which there is an appropriate $\varphi_{(k_{m_i})}^{(m_i)} \in \{\varphi_i\}_{i \in S_{kn}^{(m_i)}}$, but we may choose $S_{kn}^{(m_1)}$ so that $m_1$ is minimal. Indeed simply note if $m_1 < m$ and $k_{m_1} = k_m$ then $S_{km_1}^{(m_1)} \subseteq S_{km}^{(m)}$.

Then we consider $\{\varphi_i\}_{i \in S_{kn}^{(m_1)}}$ and again apply Proposition 2.4. There exists some $\varphi_{(k_{m_2})}^{(m_2)} \in \{\varphi_i\}_{i \in S_{kn}^{(m_2)}}$, $m_2 < m_1$ such that
\[
(\{\varphi_i\}_{i \in S_{kn}^{(m_1)}} \setminus \{\varphi_{(k_{m_1})}^{(m_1)}\}) \cup \{\varphi_{(k_{m_2})}^{(m_2)}\}
\]
is linearly independent and has the same span as $\{\varphi_i\}_{i \in S_{kn}^{(m_1)}}$. Choose the smallest such $m_2$.

By continuing this process $\{m_i\}_{i=1}^r$ is a decreasing sequence which terminates with $m_r = 1$. One final application of Proposition 2.4 implies
\[
(\{\varphi_i\}_{i \in S_{kn}^{(1)}} \setminus \{\varphi_{(1)}^{(1)}\}) \cup \{\varphi_{(0)}\}
\]
is linearly independent and has the same span as $\{\varphi_i\}_{i \in S_{kn}^{(1)}}$.

Thus we can partition $\Phi \setminus \{\varphi_{(k_n)}^{(n)}\}$ into $\ell$ sets of linear independent vectors, say $\{\{\varphi_i\}_{i \in G_j}\}_{j=1}^\ell$ where $|G_j| = |F_j|$ and span($\{\varphi_i\}_{i \in G_j}$) $=$ span($\{\varphi_i\}_{i \in F_j}$) for $j = 1, \ldots, \ell - 1$. However, recalling $\varphi_{(k_n)}^{(n)} \notin$ span($\{\varphi_i\}_{i \in F_{\ell-1}}$),
\[
\{\varphi_i\}_{i \in G_j} = \{\varphi_i\}_{i \in G_{\ell-1}} \cup \{\varphi_{(k_n)}^{(n)}\}
\]
is also linearly independent with $|G_{\ell-1}| > |F_{\ell-1}|$, contradicting that $\{\{\varphi_i\}_{i \in F_j}\}_{j=1}^\ell$ was a fundamental partition.

□

Corollary 2.7 shows that in a fundamental partition, spans of specific subsets of $\{\varphi_i\}_{i \in F_j}$, $j = 1, \ldots, \ell - 1$ contain a common subspace. This will lead to so-called transversals.
Lemma 2.10. Proposition 2.4 in [6].

Clearly \( \cup = \emptyset \) and \( \cap = \emptyset \), and we have the desired \( \phi \). Setting \( T = \cup_{j=1}^{t} (U_j \cup V_j) \), \( \{\phi_i\}_{i \in T} \) is a \( t \)-transversal of \( \{\phi_i\}_{i \in F_j} \).

Corollary 2.9. Consider the family of vectors \( \Phi = \{\phi_i\}_{i=1}^{M} \) with a fundamental partition \( \{\{\phi_i\}_{i \in F_j}\}_{j=1}^{\ell} \). Fix \( t < \ell \) and choose any \( \phi_{(r)} \in \{\phi_i\}_{i \in F_r} \) where \( t < r \). Then \( \{\{\phi_i\}_{i \in F_j}\}_{j=1}^{\ell} \) contains a \( t \)-transversal \( \{\phi_i\}_{i \in t} \), where \( \phi_{(r)} \in \{\phi_i\}_{i \in s_j} \) for all \( j \in \{1, \ldots, t\} \).

Proof. Notice if \( \{\{\phi_i\}_{i \in F_j}\}_{j=1}^{\ell} \) is a fundamental partition and we remove sets \( \{\phi_i\}_{i \in F_j} \), \( j = t + 1, \ldots, r - 1, r + 1, \ldots, \ell \), then
\[
\{\{\phi_i\}_{i \in F_1}, \ldots, \{\phi_i\}_{i \in F_t}, \{\phi_i\}_{i \in F_r}\}
\]
remains a fundamental partition for the remaining vectors
\[
\Phi \setminus \{\{\phi_i\}_{i \in \cup_{j=t+1,j \neq r}}^{\ell}\}.
\]

It therefore suffices to prove the statement for \( t = \ell - 1 \), and \( r = \ell \).

Consider the sets \( S_j^{(n)} \), \( j = 1, \ldots, \ell - 1 \), \( n = 1, 2, \ldots \) as given in Corollary 2.7 where again \( S_{kn}^{(n)} \) is a largest such set for each \( n \). Notice \( \text{span}(\{\phi_i\}_{i \in S_j^{(n)}}) \subseteq \text{span}(\{\phi_i\}_{i \in S_{kn}^{(n+1)}}) \). Since we have only finitely many vectors, there exists a \( n_0 \) such that
\[
\left| S_{k^{n_0} - 1} \right| = \left| S_{k^{n_0}} \right|.
\]

Then
\[
\left| S_{k^{n_0} - 1} \right| = \left| S_{k^{n_0}} \right|, \quad j = 1, \ldots, \ell - 1.
\]

Since \( \text{span}(\{\phi_i\}_{i \in S_j^{(n)}}) \subseteq \text{span}(\{\phi_i\}_{i \in S_{kn}^{(n)}}) \) for all \( j = 1, \ldots, \ell - 1 \), we conclude
\[
\text{span}(\{\phi_i\}_{i \in S_{kn}^{(n)}}) = \text{span}(\{\phi_i\}_{i \in S_{kn}^{(n+1)}}).
\]

Clearly \( \phi_{(r)} \in \text{span}(\{\phi_i\}_{i \in S_j^{(n)}}) \) and \( S_j^{(n)} \subseteq F_j \) for all \( j = 1, \ldots, \ell - 1 \) by construction. Set
\[
T = \cup_{j=1}^{\ell-1} S_j = \cup_{j=1}^{\ell-1} S_j^{(n_0)}, \quad \text{and we have the desired } \ell - 1 \text{ transversal } \{\phi_i\}_{i \in T}.
\]

It is simple to see that given multiple \( t \)-transversals in a fundamental partition, their union is a \( t \)-transversal of the same partition; we omit the proof. In its matroid version, this is Proposition 2.4 in [6].

Lemma 2.10. Let \( \{\{\phi_i\}_{i \in F_j}\}_{j=1}^{\ell} \) be a fundamental partition of \( \Phi = \{\phi_i\}_{i=1}^{M} \). Suppose \( \{\phi_i\}_{i \in T_1} \) and \( \{\phi_i\}_{i \in T_2} \) are \( t \)-transversals of \( \{\{\phi_i\}_{i \in F_j}\}_{j=1}^{\ell} \) where \( T_1 = \cup_{j=1}^{\ell} U_j \) and \( T_2 = \cup_{j=1}^{\ell} V_j \). Setting \( T = \cup_{j=1}^{\ell} (U_j \cup V_j) \), \( \{\phi_i\}_{i \in T} \) is a \( t \)-transversal of \( \{\{\phi_i\}_{i \in F_j}\}_{j=1}^{\ell} \).
We are now ready to prove the Rado-Horn theorem.

3. Proof of Rado-Horn and its Generalizations

We begin with the original.

**Theorem 3.1.** (Rado-Horn) Consider the family of vectors $\Phi = \{\varphi_i\}_{i=1}^M$. Then the following are equivalent:

(i) The set $\{1, \ldots, M\}$ can be partitioned into sets $\{A_j\}_{j=1}^k$ such that $\{\varphi_i\}_{i \in A_j}$ is a linearly independent set for all $j = 1, 2, \ldots, k$.

(ii) For any non-empty subset $J \subseteq \{1, \ldots, M\}$, $|J|/\dim \text{span}(\{\varphi_i\}_{i \in J}) \leq k$.

**Proof.** (i $\Rightarrow$ ii). Suppose $\{A_j\}_{j=1}^k$ is a partition of $\{1, \ldots, M\}$ such that $\{\varphi_i\}_{i \in A_j}$ is a linearly independent set for all $j = 1, 2, \ldots, k$. For any $J \subseteq \{1, \ldots, M\}$, let $J_j = J \cap A_j$. Then

$$|J| = \sum_{j=1}^k |J_j| = \sum_{j=1}^k \dim \text{span}(\{\varphi_i\}_{i \in J_j}) \leq k \dim \text{span}(\{\varphi_i\}_{i \in J})$$

giving the result.

(ii $\Rightarrow$ i). We prove the contrapositive. Suppose $\Phi$ cannot be partitioned into $k$ linearly independent sets. Then for any fundamental partition $\{\{\varphi_i\}_{i \in F_j}\}_{j=1}^\ell$, we must have $\ell > k$.

By Corollary 2.9, for any $\varphi(\ell) \in F_\ell$, $\{\varphi_i\}_{i \in \ell}$ contains a $k$-transversal, $T$ with $\varphi(\ell) \in \text{span}(\{\varphi_i\}_{i \in T})$. Then we have

$$\frac{|T \cup \{(\ell)\}|}{\dim \text{span}(\{\varphi_i\}_{i \in T \cup \{(\ell)\}})} = \frac{k + 1}{\dim \text{span}(\{\varphi_i\}_{i \in T})} > k.$$  \hfill \Box

One of the benefits of this proof is that the ideas generalize to many other versions of the Rado-Horn theorem. It is a simple matter to adapt the ideas of this proof to show the following generalized version of the Rado-Horn theorem which originally appeared in [4]. We omit the details as the ideas are similar to the previous proof.

**Theorem 3.2.** (Generalized Rado-Horn) Consider the family of vectors $\Phi = \{\varphi_i\}_{i=1}^M$. Then the following are equivalent.

(i) There exists a subset $H \subseteq \{1, \ldots, M\}$ such that $\{\varphi_i\}_{i \in H}$ can be partitioned into $k$ linearly independent sets.

(ii) For any non-empty subset $J \subseteq \{1, \ldots, M\}$, we have $(|J| - |H|)/\dim \text{span}(\{\varphi_i\}_{i \in J}) \leq k$.

Transversals in a fundamental partition also explain why the Rado-Horn inequality can fail when $\Phi$ cannot be partitioned into $k$ linearly independent sets. The following redundant version of Rado-Horn was originally proven in [4].

**Theorem 3.3.** (Redundant Rado-Horn) Consider the family of vectors $\Phi = \{\varphi_i\}_{i=1}^M$ in a vector space $V$. If this set cannot be partitioned into $k$ linearly independent sets, then there exists a partition $\{A_j\}_{j=1}^k$ of $\{1, \ldots, M\}$ and a subspace $S$ of $V$ such that the following hold:

(i) For all $1 \leq j \leq k$, there exists a subset $S_j \subseteq A_j$ such that $S = \text{span}(\{\varphi_i\}_{i \in S_j})$.

(ii) For $J = \{i : \varphi_i \in S\}$, $|J|/\dim \text{span}(\{\varphi_i\}_{i \in J}) > k$.
(iii) For all $1 \leq j \leq k$, $\{\varphi_i\}_{i \in A_j \setminus S_j}$ is linearly independent.

**Proof.** Take a fundamental partition $\{\{\varphi_i\}_{i \in F_j}\}_{j=1}^\ell$ of $\Phi$, and consider the partition $\{A_j\}_{j=1}^k = \{F_1, \ldots, F_k-1, \cup_{r=k}^{\ell} F_r\}$. We will show there exists a subspace $S$ which satisfies (i), (ii), and (iii) for $\{\{\varphi_i\}_{i \in A_j}\}_{j=1}^k$.

By Corollary 2.9, for each $r \in F_j$, $j = k+1, \ldots, \ell$, there exists a $k$-transversal, say $\{\varphi_i\}_{i \in T_r}$, of $\{\{\varphi_i\}_{i \in F_j}\}_{j=1}^\ell$ containing $\varphi_r$ in span($\{\varphi_i\}_{i \in T_r}$). By Lemma 2.10, the union

$$T = \cup_{\{r: \varphi_r \in F_j, j=k+1, \ldots, \ell\}} T_r$$

gives us that $\{\varphi_i\}_{i \in T}$ is a $k$-transversal of $\{\{\varphi_i\}_{i \in F_j}\}_{j=1}^\ell$ which satisfies $\varphi_r \in \text{span}(\{\varphi_i\}_{i \in T})$ for all $\varphi_r \in F_j$, $j = k+1, \ldots, \ell$. Thus

$$\text{span}(\{\varphi_i\}_{i \in F_j}) \subseteq \text{span}(\{\varphi_i\}_{i \in T})$$

for all $j = k+1, \ldots, \ell$.

Finally, set $S = \text{span}(\{\varphi_i\}_{i \in T})$ and $S_i = T \cap F_i$ for $i = 1, \ldots, k-1$ with $S_k = T \cap (\cup_{r=k}^{\ell} F_r)$.

Then (i) and (ii) follow since $\{\varphi_i\}_{i \in T}$ is a $k$-transversal which contains in its span at least one $\varphi \in \{\varphi_i\}_{i \in F_j}, j > k$ (in this case all of them). Clearly for $j = 1, \ldots, k-1$, $\{\varphi_i\}_{i \in A_j \setminus S_j} \subseteq \{\varphi_i\}_{i \in F_j}$ is linearly independent. Lastly by the way we constructed our transversal,

$$\{\varphi_i\}_{i \in A_k \setminus S_k} \subseteq \{\varphi_i\}_{i \in (\cup_{j=k}^{\ell} F_j) \setminus (\cup_{j=k}^{\ell} F_j)} \subseteq \{\varphi_i\}_{i \in F_k},$$

which is also linearly independent. \qed

### 4. Constructing a Fundamental Partition

Our proof of the Rado-Horn theorem relies only on the existence of a fundamental partition. Interestingly, we can build a fundamental partition where we use Rado-Horn as a tool in the construction. This process is much like finding the so-called flag transversal for a sum of matroids [9]. It will be helpful to define the concept of a quasi-transversal which, like the transversal, is inspired from a matroid version [8].

**Definition 4.1.** Given a fundamental partition $\{\{\varphi_i\}_{i \in F_j}\}_{j=1}^\ell$ of $\Phi = \{\varphi_i\}_{i=1}^M$, let $t \leq \ell$ and $T \subseteq \{1, \ldots, M\}$. We call $\{\varphi_i\}_{i \in T}$ a $t$-quasi-transversal of $\{\{\varphi_i\}_{i \in F_j}\}_{j=1}^\ell$ if $T = \cup_{j=1}^t S_j$ where $S_j \subseteq F_j$ and

$$\text{span}(\{\varphi_i\}_{i \in S_j}) = \text{span}(\{\varphi_i\}_{i \in S_k}) \quad j, k \in \{1, \ldots, t - 1\}$$

$$\text{span}(\{\varphi_i\}_{i \in S_j}) \subseteq \text{span}(\{\varphi_i\}_{i \in S_k}) \quad j \in \{1, \ldots, t - 1\}.$$

Quasi-transversals will form building blocks for our construction of a fundamental partition. Initially this is a problem since quasi-transversals are defined in terms of existing fundamental partitions. In order to proceed, we must find vectors which form a quasi-transversal in some yet unknown fundamental partition. Choosing vectors which maximize the Rado-Horn inequality is a reasonable starting point.

**Proposition 4.2.** Given a family of vectors $\Phi = \{\varphi_i\}_{i=1}^M$, suppose $J \subseteq \{1, \ldots, M\}$ maximizes $|J|/\text{dim span}(\{\varphi_i\}_{i \in J})$. Then in a fundamental partition of $\{\varphi_i\}_{i \in J}$, say $\{\{\varphi_i\}_{i \in F_j}\}_{j=1}^\ell$, $\{\varphi_i\}_{i \in J}$ is an $\ell$-quasi-transversal.
Proof. By Corollary 2.9 and Lemma 2.10, \( \{\{\varphi_i\}_{i \in F_j}\}_{j=1}^\ell \) contains a maximal \((\ell-1)\)-transversal, \( \{\varphi_i\}_{i \in T} \), where \( \text{span}(\{\varphi_i\}_{i \in F_j}) \subseteq \text{span}(\{\varphi_i\}_{i \in T}) \). Define the set \( T' = T \cup F'_\ell \) so that \( \{\varphi_i\}_{i \in T'} \) is an \( \ell \)-quasi-transversal of \( \{\{\varphi_i\}_{i \in F'_j}\}_{j=1}^\ell \). For contradiction, suppose \( \{\varphi_i\}_{i \in J} \) was not an \( \ell \)-quasi-transversal of \( \{\{\varphi_i\}_{i \in F'_j}\}_{j=1}^\ell \), then \( \{\varphi_i\}_{i \in J \setminus T'} \) cannot be an \((\ell-1)\)-transversal in \( \{\{\varphi_i\}_{i \in F'_j}\}_{j=1}^\ell \). Specifically,

\[
|F'_{\ell-1} \setminus T'| < |F'_1 \setminus T'| = \dim \text{span}(\{\varphi_i\}_{i \in J}) - \dim \text{span}(\{\varphi_i\}_{i \in T'}),
\]

which then implies

(2) \(|J| - |T'| = |J \setminus T'| < \sum_{j=1}^{\ell-1} |F'_j \setminus T'| < (\ell - 1)[\dim \text{span}(\{\varphi_i\}_{i \in J}) - \dim \text{span}(\{\varphi_i\}_{i \in T'})].
\]

Using (2) and that \( \{\varphi_i\}_{i \in T'} \) is an \( \ell \)-quasi-transversal of \( \{\{\varphi_i\}_{i \in F'_j}\}_{j=1}^\ell \), we have

\[
\frac{|J| - |T'|}{\dim \text{span}(\{\varphi_i\}_{i \in J}) - \dim \text{span}(\{\varphi_i\}_{i \in T'})} < \ell - 1 < \frac{|T'|}{\dim \text{span}(\{\varphi_i\}_{i \in T'})}.
\]

It now follows that

\[
|T'| \dim \text{span}(\{\varphi_i\}_{i \in J}) = |T'| [\dim \text{span}(\{\varphi_i\}_{i \in T'}) + (\dim \text{span}(\{\varphi_i\}_{i \in J}) - \dim \text{span}(\{\varphi_i\}_{i \in T'}))] > |T'| \dim \text{span}(\{\varphi_i\}_{i \in T'}) + (|J| - |T'|) \dim \text{span}(\{\varphi_i\}_{i \in T'}) = |J| \dim \text{span}(\{\varphi_i\}_{i \in T'}),
\]

giving \(|T'| / \dim \text{span}(\{\varphi_i\}_{i \in T'}) > |J| / \dim \text{span}(\{\varphi_i\}_{i \in J})\), a contradiction. \( \square \)

Proposition 4.2 is not adequate since it says nothing about being a quasi-trasversal in a fundamental partition of the entire family \( \Phi \). By picking a slightly different \( J \subseteq \{1, \ldots, M\} \), we can find the needed quasi-transversals.

Lemma 4.3. Suppose \( \Phi = \{\varphi_i\}_{i=1}^M \) can be partitioned into at fewest \( \ell \) linearly independent sets. Let \( K \subseteq \{1, \ldots, M\} \) be such that

\[
\frac{|K|}{\dim \text{span}(\{\varphi_i\}_{i \in K})} = \ell - 1,
\]

and for any other set \( L \) satisfying this equality,

\[
|\{i : \varphi_i \in \text{span}(\{\varphi_i\}_{i \in K}) \setminus K\}| \geq |\{i : \varphi_i \in \text{span}(\{\varphi_i\}_{i \in L}) \setminus L\}|.
\]

Let \( J = \{i : \varphi_i \in \text{span}(\{\varphi_i\}_{i \in K})\} \). Then for any fundamental partition \( \{\{\varphi_i\}_{i \in F_j}\}_{j=1}^\ell \) of \( \Phi \), \( \{\varphi_i\}_{i \in J} \) is an \( \ell \)-quasi-transversal.

Proof. First note such a set \( K \neq \emptyset \) since an \((\ell - 1)\)-transversal in a fundamental partition, which we have by Corollary 2.9, satisfies the equality.

With \( J \) now chosen, let \( \{\{\varphi_i\}_{i \in F_j}\}_{j=1}^\ell \) be any fundamental partition of \( \Phi \), and consider the sets \( J \cap F_j \). We must have

(3) \(|J \cap F_\ell| \leq |J \setminus K|\)
for otherwise we could find a maximal \((\ell - 1)\)-transversal \(\{\varphi_i\}_{i \in L}\) as a consequence of Corollary 2.9 and Lemma 2.10. This would imply
\[
\frac{|L|}{\dim \text{ span}(\{\varphi_i\}_{i \in L})} = \ell - 1,
\]
and
\[
|\{i : \varphi_i \in \text{ span}(\{\varphi_i\}_{i \in L})\} \setminus L| \geq |J \cap F_\ell| > |J \setminus K| = |\{i : \varphi_i \in \text{ span}(\{\varphi_i\}_{i \in K})\} \setminus K|
\]
a contradiction.

With (3) in mind, notice
\[
|J \cap F_j| \leq \dim \text{ span}(\{\varphi_i\}_{i \in J}) = \dim \text{ span}(\{\varphi_i\}_{i \in K}),
\]
but suppose the inequality was strict for some \(j \in \{1, \ldots, \ell - 1\}\). Then we have
\[
|K| + |J \setminus K| = |J| = \sum_{j=1}^{\ell} |J \cap F_j| = \sum_{j=1}^{\ell-1} |J \cap F_j| + |J \cap F_\ell| < (\ell - 1) \dim \text{ span}(\{\varphi_i\}_{i \in K}) + |J \cap F_\ell| = |K| + |J \cap F_\ell|,
\]
and we see \(|J \cap F_\ell| > |J \setminus K|\), a contradiction.

We conclude
\[
|J \cap F_j| = \dim \text{ span}(\{\varphi_i\}_{i \in J})
\]
for all \(j \in \{1, \ldots, \ell - 1\}\) and
\[
|J \cap F_\ell| = |F_\ell|.
\]
It follows that \(\{\varphi_i\}_{i \in J}\) is an \(\ell\)-quasi-transversal of \(\{\{\varphi_i\}_{i \in F_j}\}_{j=1}^\ell\).

**Definition 4.4.** Given a family of vectors \(\Phi = \{\varphi_i\}_{i=1}^M\) which can be partitioned into at fewest \(\ell\) linearly independent sets, let \(J \subset \{1, \ldots, M\}\) be given as in Lemma 4.3. Then we will say \(\{\varphi_i\}_{i \in J}\) is a **universal quasi-transversal** of \(\Phi\).

Notice that for any fundamental partition \(\{\{\varphi_i\}_{i \in F_j}\}_{j=1}^\ell\) of \(\{\varphi_i\}_{i=1}^M\), a universal quasi-transversal \(\{\varphi_i\}_{i \in J}\) must be an \(\ell\)-quasi-transversal with \(F_\ell \subseteq J\).

Now that we have a quasi-transversal for some fundamental partition, albeit unknown, the next two results show projecting onto the orthogonal complements of the spans of such transversals does not greatly effect the structure of the partition. Theorem 4.6 is the main result needed for our construction.
Lemma 4.5. Consider family of vectors $\Phi = \{\varphi_i\}_{i=1}^M$ and suppose $\{\varphi_i\}_{i \in T}$ is an $\ell$-quasi-transversal of the fundamental partition $\{\{\varphi_i\}_{i \in F_j}\}_{j=1}^\ell$ which satisfies $F_\ell \subseteq T$. Let $P_T$ be the orthogonal projection onto $\text{span}\{\varphi_i\}_{i \in T}$ and suppose $F'_j = \{i : i \in F_j \setminus T\}$. Then $\{(I - P_T)\varphi_i\}_{i \in F'_j}$ is a fundamental partition of $\{(I - P_T)\varphi_i\}_{i \in T}$ where

$$\ell' = \max\{j : F_j \neq F'_j\}.$$  

Proof. Note the family $\{(I - P_T)\varphi_i\}_{i \in T}$ is precisely the non-zero elements of $\Phi$ under the projection $P_T$.

We first show $\{(I - P_T)\varphi_i\}_{i \in F'_j}$ is linearly independent for $j = 1, \ldots, \ell'$. Indeed suppose there exists scalars $\{a_i\}_{i \in F'_j}$ such that $\sum_{i \in F'_j} a_i(I - P_T)\varphi_i = 0$. Then $\sum_{i \in F'_j} a_i\varphi_i \in \text{span}\{\varphi_i\}_{i \in F_j \setminus F'_j}$ where $\{\varphi_i\}_{i \in F_j \setminus F'_j}$ is non-empty due to (4). Since $\{(I - P_T)\varphi_i\}_{i \in F'_j}$ is linearly independent, $a_i = 0$ for all $i \in F'_j$.

Now suppose these independent sets do not form a fundamental partition. Then there exists some other partition of $\{1, \ldots, M\} \setminus T$, say $\{A_j\}_{j=1}^s$ such that $\{(I - P_T)\varphi_i\}_{i \in A_j}$ is linearly independent for all $j = 1, \ldots, s$ and there is some $k < \ell'$ such that $|A_k| > |F'_k|$ but $|A_j| = |F'_k|$ for all $j < k$. It now suffices to show $\{(I - P_T)\varphi_i\}_{i \in (F'_j \cup A_j)}$ is linearly independent for $j = 1, \ldots, k$, for this would contradict that $\{(I - P_T)\varphi_i\}_{i \in F'_j}$ was a fundamental partition.

For scalars $a_i$, consider $\sum_{i \in (F'_j \cup A_j)} a_i\varphi_i = 0$. The projection $I - P_T$, this becomes

$$\sum_{i \in (F'_j \cup A_j)} a_i(I - P_T)\varphi_i = \sum_{i \in A_j} a_i(I - P_T)\varphi_i = 0,$$

and $a_i = 0$ for $i \in A_j$. But then

$$\sum_{i \in (F'_j \cup A_j)} a_i\varphi_i = \sum_{i \in A_j} a_i\varphi_i = 0,$$

and $a_i = 0$ for all $i \in (F'_j \cup A_j)$. \hfill \Box

Theorem 4.6. Suppose the family of vectors $\Phi = \{\varphi_i\}_{i=1}^M$ can be partitioned into at least $\ell$ linearly independent sets. Let $\{\varphi_i\}_{i \in J}$ be a universal quasi-transversal of $\Phi$, and let $P_J$ be the orthogonal projection onto $\text{span}\{\varphi_i\}_{i \in J}$. Assuming $J \neq \{1, \ldots, M\}$, let $\{\{\varphi_i\}_{i \in G_j}\}_{j=1}^\ell$ and $\{\{(I - P_J)\varphi_i\}_{i \in G'_j}\}_{j=1}^\ell$ be fundamental partitions of $\{\varphi_i\}_{i \in J}$ and $\{(I - P_J)\varphi_i\}_{i \in \{1, \ldots, M\} \setminus J}$ respectively. Then $\ell' \leq \ell$ and $\{\{\varphi_i\}_{i \in G_j \cup G'_j}\}_{j=1}^\ell$ is a fundamental partition of $\Phi$ where we use the convention $G'_j = \emptyset$ for $\ell' < j \leq \ell$.

Proof. First note $\{\varphi_i\}_{i \in G'_j}$ are not all empty since $J \subset \{1, \ldots, M\}$. Also $\{\varphi_i\}_{i \in G_j\}_{j=1}^\ell$ must be an $\ell$-quasi-transversal of $\{\varphi_i\}_{i \in J}$.

We will show $\{\varphi_i\}_{i \in G_j \cup G'_j}$ are linearly independent for $j \in \{1, \ldots, \ell\}$. Suppose

$$\sum_{i \in G_j \cup G'_j} a_i\varphi_i = \sum_{i \in G_j} a_i\varphi_i + \sum_{i \in G'_j} a_i\varphi_i = 0.$$
Under the projection \((I - P_J)\), this becomes
\[
\sum_{i \in G_j'} a_i(I - P_J)\varphi_i = 0,
\]
and \(a_i = 0\) for \(i \in G_j'\) since \(\{(I - P_J)\varphi_i\}_{i \in G_j'}^r\) is a fundamental partition. Then
\[
\sum_{i \in G_j \cup G_j'} a_i\varphi_i = \sum_{i \in G_j} a_i\varphi_i = 0,
\]
but \(\{(\varphi_i)_{i \in G_j'}\}_{j=1}^l\) is also a fundamental partition. We conclude \(\{\varphi_i\}_{i \in G_j \cup G_j'}\) is a linearly independent set for \(j \in \{1, \ldots, \ell\}\).

Now that we have linear independence of these sets, we will show \(\{(\varphi_i)_{i \in G_j \cup G_j'}\}_{j=1}^l\) forms a fundamental partition of \(\Phi\). This will automatically imply \(\ell' \leq \ell\). For contradiction, suppose this was not the case. Then there exists a fundamental partition \(\{(\varphi_i)_{i \in F_j'}\}_{j=1}^\ell\) such that for some \(1 \leq t < \ell\), \(|F_j| = |G_j \cup G_j'|\) for \(j < t\) but \(|F_t| > |G_t \cup G_t'|\). We define
\[
F_j' = \{i \in F_j : (I - P_J)\varphi_i \neq 0\}
\]
and compare \(\{(I - P_J)\varphi_i\}_{i \in F_j'}\) with \(\{(I - P_J)\varphi_i\}_{i \in G_j'}\).

Since \(J\) makes up a universal quasi-transversal, Lemma 4.5 gives us that \(\{(I - P_J)\varphi_i\}_{i \in F_j'}\) is a fundamental partition of \(\{(I - P_J)\varphi_i\}_{i \notin J}\), and by hypothesis, so is \(\{(I - P_J)\varphi_i\}_{i \in G_j'}\). Hence
\[
|F_j'| = |G_j'|
\]
for \(j \in \{1, \ldots, \ell\}\). However \(|F_t| > |G_t \cup G_t'|\) requires
\[
|F_t'| > |G_t'|
\]
a contradiction. \(\square\)

We can now construct a fundamental partition by repeated application of Theorem 4.6.

4.1. Construction of a Fundamental Partition. Let \(\Phi = \{\varphi_i\}_{i=1}^M = \{\varphi_{1i}\}_{i=1}^M\) be a family of vectors where we have added the extra index in order to track an iterative process of projections. Suppose
\[
\max_{J \subseteq \{1, \ldots, M\}} \left[ \frac{|J|}{\dim \operatorname{span}(\{\varphi_i\}_{i \in J})} \right] = k_1.
\]
Then a fundamental partition of \(\Phi\) contains \(k_1\) linearly independent sets, and we may find a universal quasi-transversal of \(\Phi\) by searching through \(K \subseteq \{1, \ldots, M\}\) such that \(|K| / \dim \operatorname{span}(\{\varphi_i\}_{i \in K}) = k_1 - 1\). Choose \(T_1 \subseteq \{1, \ldots, M\}\) so that \(\{\varphi_i\}_{i \in T_1}\) comprises such a universal quasi-transversal. Let
\[
t_1 = \dim \operatorname{span}(\{\varphi_{1i}\}_{i \in T_1}),
\]
\[
s_1 = |T_1| - (k_1 - 1)t_1.
\]
Then we know exactly how this quasi-transversal appears in a fundamental partition. It is not difficult to see that we may partition \(T_1\) as \(\{T_{1j}\}_{j=1}^{k_1}\) where
(i) \(|T_{1j}| = t_1, j = 1, \ldots, k_1 - 1\)
(ii) $|T_{1j}| = s_1$, $j = k_1$
(iii) $\text{span}(\{\varphi_i\}_{i \in T_{1n}}) = \text{span}(\{\varphi_i\}_{i \in T_{1m}})$, $n, m \neq k_1$
(iv) $\text{span}(\{\varphi_i\}_{i \in T_{1k_1}}) \subseteq \text{span}(\{\varphi_i\}_{i \in T_{1j}})$, $j = 1, \ldots, k_1 - 1$.

Let $P_{T_1}$ be the orthogonal projection of $\Phi$ onto $\text{span}(\{\varphi_{1i}\}_{i \in T_1})$. Define $\Phi_2 = \{(I - P_{T_1})\varphi_{1i}\}_{i \in T_1} = \{\varphi_{2i}\}_{i \in T_1}$. Finding a fundamental partition of $\Phi_2$ will give us a fundamental partition of $\Phi$ by Theorem 4.6.

Examine subsets of the indices $\{1, \ldots, M\} \setminus T_1$ so that

$$\max_{J \subseteq \{1, \ldots, M\} \setminus T_1} \left[ \frac{|J|}{\dim \text{span}(\{\varphi_{2i}\}_{i \in J})} \right] = k_2.$$

We now know a fundamental partition of $\Phi_2$ contains $k_2$ linearly independent sets, and we may again find a universal quasi-transversal. Choose $T_2 \subseteq \{1, \ldots, M\} \setminus T_1$ so that $\{\varphi_{2i}\}_{i \in T_2}$ comprises a universal quasi-transversal, and let

$$t_2 = \dim \text{span}(\{\varphi_{2i}\}_{i \in T_2}),$$
$$s_2 = |T_2| - (k_2 - 1)t_2.$$

We may partition $T_2$ as $\{T_{2j}\}_{j=1}^{k_2}$ where

(i) $|T_{2j}| = t_2$, $j = 1, \ldots, k_2 - 1$
(ii) $|T_{2j}| = s_2$, $j = k_2$
(iii) $\text{span}(\{\varphi_i\}_{i \in T_{2n}}) = \text{span}(\{\varphi_i\}_{i \in T_{2m}})$, $n, m \neq k_2$
(iv) $\text{span}(\{\varphi_i\}_{i \in T_{2k_2}}) \subseteq \text{span}(\{\varphi_i\}_{i \in T_{2j}})$, $j = 1, \ldots, k_2 - 1$.

We continue so that $P_{T_j}$ is the orthogonal projection of $\Phi_r$ onto $\text{span}(\{\varphi_{ri}\}_{i \in T_r})$. Define $\Phi_{r+1} = \{(I - P_{T_r})\varphi_{ri}\}_{i \in T_1 \cup \ldots \cup T_r} = \{\varphi_{(r+1)i}\}_{i \in T_1 \cup \ldots \cup T_r}$. Examine subsets of the indices $\{1, \ldots, M\} \setminus (\cup_{j=1}^{r} T_j)$ so that

$$\max_{J \subseteq \{1, \ldots, M\} \setminus (\cup_{j=1}^{r} T_j)} \left[ \frac{|J|}{\dim \text{span}(\{\varphi_{(r+1)i}\}_{i \in J})} \right] = k_{r+1}.$$

Now choose $T_{r+1} \subseteq \{1, \ldots, M\} \setminus (\cup_{j=1}^{r} T_j)$ so that $\{\varphi_{(r+1)i}\}_{i \in T_{r+1}}$ is a universal quasi-transversal in $\Phi_{r+1}$. Letting

$$t_{r+1} = \dim \text{span}(\{\varphi_{(r+1)i}\}_{i \in T_{r+1}}),$$
$$s_{r+1} = |T_{r+1}| - (k_{r+1} - 1)t_{r+1},$$

we may partition $T_{r+1}$ as $\{T_{(r+1)j}\}_{j=1}^{k_{r+1}}$

(i) $|T_{(r+1)j}| = t_{r+1}$, $j = 1, \ldots, k_{r+1} - 1$
(ii) $|T_{(r+1)j}| = s_{r+1}$, $j = k_{r+1}$
(iii) $\text{span}(\{\varphi_i\}_{i \in T_{(r+1)n}}) = \text{span}(\{\varphi_i\}_{i \in T_{(r+1)m}})$, $n, m \neq k_{r+1}$
(iv) $\text{span}(\{\varphi_i\}_{i \in T_{(r+1)k_{r+1}}}) \subseteq \text{span}(\{\varphi_i\}_{i \in T_{(r+1)j}})$, $j = 1, \ldots, k_{r+1} - 1$.

Notice $k_r > k_{r+1}$. At some point, we will have used up all our indices. To be precise, this occurs after $z$ iterations where $k_z \neq 0$ but $k_{z+1} = 0$. Finally, for $j > k_r$ adopt the convention $T_{rj} = \emptyset$. Then letting

$$F_j = \cup_{r=1}^{z} T_{rj}, \quad j = 1, \ldots, k_1,$$

$\{\varphi_i\}_{i \in F_j}_{j=1}^{z}$ is a fundamental partition of $\Phi$. AN ELEMENTARY, ILLUSTRATIVE PROOF OF THE RADO-HORN THEOREM 13
We have constructed a fundamental partition by repeatedly finding universal quasi-transversals and applying Theorem 4.6. Figure 2 provides an example of a fundamental partition showing values $t_i, k_i, s_i, i = 1, \ldots, z$ where $z = 3$.

**Figure 2.** Fundamental partition constructed from quasi-transversals of appropriate projections

**Remark 4.7.** We have essentially used Rado-Horn and transversals to describe many of the spanning properties of the vectors. For example, using the notation from the above construction, a family of vectors $\Phi = \{\varphi_i\}_{i=1}^M$ spans a $(\sum_{i=1}^z t_i)$-dimensional space and can be partitioned into at most $k_z$ spanning sets when $t_z = s_z$ and at most $k_z - 1$ spanning sets when $t_z > s_z$.

**Acknowledgment:** The authors thank the referee for the helpful suggestions and bringing our attention to the paper [9].

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