Extensions of pure states and the Laurent operator

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Chapter 1

Introduction

1.1 Overview

Chapter 1 introduces the main concepts and notation. In chapter 2 there is nothing new; we have brought together some known characterizations of the pure states \( P(\mathbb{D}) \) of \( \mathbb{D} \): \( P(\mathbb{D}) \) is the Stone-Cech compactification of the natural numbers \( \mathbb{N} \), and there exists a one-to-one correspondence between members of \( P(\mathbb{D}) \) and ultra-filters over \( \mathbb{N} \). If \( \mathbb{D} \) is the diagonal algebra of \( B(l^2(\mathbb{N})) \), then everything in chapter 2 holds with \( \mathbb{N} \) replaced by \( \mathbb{Z} \).

In 2.3 we make use of an argument of Kadison and Singer to give a necessary and sufficient condition for a pure state \( f \in P(\mathbb{D}) \) to have a unique extension in terms of the ultra-filter \( \mathcal{U}_f \) that corresponds to \( f \). For each \( f \in P(\mathbb{D}) \) and \( T \in B(l^2(\mathbb{N})) \) we define:

\[
\beta_f(T) = \inf \{ \|P_\sigma(T - E(T))P_\sigma\| : \sigma \in \mathcal{U}_f \}
\]

and show that \( \beta_f(T) = 0 \) for every \( T \in B(l^2(\mathbb{N})) \) if and only if \( f \) has a unique extension to a pure state of \( B(l^2(\mathbb{N})) \). A similar result holds for \( B(l^2(\mathbb{Z})) \). As a corollary we will have proved that the non-singular pure states of \( \mathbb{D} \) do have a unique extension to pure states of \( B(l^2(\mathbb{N})) \), a fact that is known already.

In 2.4 we give a modest partial answer to a conjecture of Glimm, who in [9] conjectured that if \( A \) is a proper \( C^* \) sub-algebra of \( C^* \) algebra \( B \), then there exist distinct pure states on \( B \) whose restriction to \( A \) coincide. In other words, \( C^* \) sub-algebras do not separate the pure states of \( C^* \)-algebras that strictly contain them. We will show that Glimm’s conjecture holds in the special case when \( A \) is abelian. This however does not provide a negative answer to the KS problem, because there are always pure states on the larger algebra whose restriction to the smaller one is not a pure state.

The third chapter deals with the Laurent operator. The two main results are theorems 3.2.1, 3.2.3. The first claims that if \( \varphi \in L^\infty(0, 2\pi) \) is real, and \( \varepsilon > 0 \), and \( H(\varphi, \varepsilon) \) is the family of sets \( \Lambda \subset \mathbb{Z} \) for which \( |P_\Lambda(L_\varphi - \varphi)| \)

\[\ldots\]
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\[ \varphi(0)P\Lambda \leq \varepsilon \] then \( H(\varphi, \varepsilon) \) contains a set \( \Lambda \subset \mathbb{Z} \) of positive density, that is \( \lim_{N \to \infty} \frac{|\Lambda \cap [1, N]|}{N} > 0 \), and the second characterizes those Laurent operators having the paving property: \( L_\varphi \) has the paving property if and only if \( H(\varphi, \varepsilon) \) contains a syndetic subset of \( \mathbb{Z} \). (A set \( \Lambda \subset \mathbb{Z} \) is syndetic if \( \mathbb{Z} \) can be covered by a finite number of translations of \( \Lambda \)). Thanks to the invariance of \( H(\varphi, \varepsilon) \) under translations, this characterization says that one can pave \( L_\varphi \) if and only if one can pave it by translations of a certain syndetic set.

The general method in chapter 3 is to consider \( H(\varphi, \varepsilon) \) as a subset of \( Y = \{0, 1\}^\mathbb{Z} \), which together with the translation \( (T_x(n)) = x(n + 1) \), becomes a dynamical topological system. It turns out that \( H(\varphi, \varepsilon) \) is a translation-invariant compact subset of \( Y \) (in the product topology), which thanks to results from local Banach Space theory and dynamical topological system theory, namely, results by Bourgain-Tzafriri and Rusza, is rich enough to contain sets of positive density.

The last chapter collects several equivalent formulations of the Kadison-Singer problem. We have added two new ones. Using an argument due to Keith Ball ([3]) we can show that the following formulation is equivalent to the KS problem.

**Restricted Invertibility.** There exists a universal constant \( c > 0 \) and a natural number \( m \) such that for every linear \( T : l^2_n \to l^2_n \) satisfying \( \|T e_i\| = 1 \) for each \( i = 1, 2, \ldots, n \), there exists a partition \( \sigma_1, \ldots, \sigma_m \) of \( \{1, 2, \ldots, n\} \) such that

\[ \|TP_{\sigma_i} x\| \geq c\|P_{\sigma_i} x\| \]

In chapter 4 we will show that restricted invertibility is equivalent to the paving property:

**Paving Property.** Every operator \( T \in B(l^2(\mathbb{N})) \) has the following property, called the paving property:

\[ \inf \{\|\sum_{j=1}^{n} P_j(T - E(T))P_j\| : \sum_{j=1}^{n} P_j = Id, \ P_j \perp P_i \forall i \neq j\} = 0 \quad (1.1) \]

All vector spaces in this work are assumed to be over the field of complex numbers.

### 1.2 The Kadison-Singer Problem

Kadison and Singer were interested in the uniqueness of extension of homomorphisms defined on maximal abelian \( C^* \) - sub-algebras of \( B(H) \) (the space of bounded operators on a separable Hilbert space \( H \)), to positive functionals of the algebra \( B(H) \). A homomorphism of a \( C^* \)-algebra \( A \) is a linear and multiplicative function \( h \), satisfying \( h(x) = h(x^*) \) for each \( x \in A \). In particular, \( f(e) = 1 \), where \( e \) is the unit of \( A \). A positive functional on
1.3. PURE STATES IN QUANTUM MECHANICS

A $C^*$-algebra $A$ is a linear functional $f$ satisfying $f(e) \neq 0$ and $f(x^*x) > 0$ for each $x \in A$. The following problem was raised by Kadison and Singer in 1959 [14].

**Problem 1.2.1.** Let $A$ be a maximal abelian $C^*$ sub-algebra of $B(H)$. Does every homomorphism of $A$ have a unique extension to a positive functional of $B(H)$?

As we shall see in the next chapter, the problem of uniqueness is the much harder one; existence is guaranteed.

Kadison and Singer showed that in general the answer is negative. They proved that if $A$ is isomorphic to $L^\infty(0,1)$ then there are homomorphisms of $A$ that have more than one extension to a positive functional of $B(H)$. They also considered the case where $A$ is isomorphic to $l^\infty(\mathbb{N})$ (or $l^\infty(\mathbb{Z})$), and in that case they did not settle the uniqueness problem.

The algebra $l^\infty(\mathbb{N})$ is isomorphic to the diagonal algebra $D \subset B(l^2(\mathbb{N}))$, of all operators having the standard basis vectors as eigenvectors. The members of $D$ are called diagonal operators. For an operator $D \in D$, the entries $d_{m,n} = \langle De_n, e_m \rangle$ will vanish for $n \neq m$. The homomorphisms of $D$ are well known. For example, for each $n \in \mathbb{N}$ the functional $\omega_n(D) = \langle De_n, e_n \rangle$ is a homomorphism of $D$, and every homomorphism $\rho$ on $D$ is given by $\rho(D) = \lim U \langle De_n, e_n \rangle$, where $U$ is an ultra-filter above $\mathbb{N}$, determined by $\rho$. We shall discuss the homomorphisms of $D$ in more detail in the next chapter.

One way to obtain extensions of homomorphisms of $D$, is to look at the map $E : B(l^2(\mathbb{N})) \to D$, mapping every operator $T$ to the diagonal operator $E(T)$ defined by: $\langle E(T)e_n, e_m \rangle = \delta_{nm} \langle Te_n, e_m \rangle$ where $\delta_{nm} = 1$ if $n = m$, and zero otherwise. $E$ is an order preserving projection of norm 1, and for every homomorphism $\rho$ on $D$, $\rho E$ is a positive functional of $B(l^2(\mathbb{N}))$. Of course, the question is whether this is the only way to obtain homomorphisms of $D$.

1.3 Pure states in Quantum Mechanics

It is well known that the set of homomorphisms of an abelian $C^*$-algebra is precisely the set of extreme points among the positive functionals on that algebra, normalized by $f(e) = 1$. That the normalized positive functionals do have extreme points follows from the fact that they form a convex and $w^*$-compact subset of the dual Banach-space of the $C^*$-algebra.

Normalized positive functionals on a $C^*$-algebra are called states, and the extreme points among the states are called pure states. The choice of these names has to do with a physical interpretation of normalized positive functionals.

Physicists attach to certain physical systems in quantum mechanics a separable Hilbert space $H$ in such a way as to establish a one-to-one correspondence between observables of the system, (that is - those physical
quantities of the system that can be measured, like momentum and energy) and self-adjoint operators on $H$, and another one-to-one correspondence between states of the system and bounded, positive operators on $H$ whose trace is equal to 1. These latter operators are called density operators. The correspondence is such that given a Borel-measurable set $\Delta \in \mathbb{R}$, the probability $p(\Delta, T)$ that a measurement of an observable corresponding to an operator $T$ when the system is in a state corresponding to a density-operator $U$, will belong to the set $\Delta$, is equal to

$$p(\Delta, T) = \text{tr}(P^T_\Delta U) = \sum_{n=1}^{\infty} \langle P^T_\Delta U e_n, e_n \rangle$$  \hspace{1cm} (1.2)$$

where $P^T_\Delta$ is the spectral projection that corresponds to $\Delta, T$, and $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis in $H$. A state of the system for which the corresponding density-operator is a projection is called a pure state. If $U$ is a projection, then in order that it’s trace will be equal to 1 it needs to be a one-dimensional projection, so that $U = x \otimes x$ for some unit vector $x \in H$. Recall that $x \otimes x$ is a one-dimensional projection on the span of $x$ defined by: $x \otimes x(y) = \langle y, x \rangle x$. In a pure state, the probability of equation (1.2) takes the form:

$$p(\Delta, T) = \text{tr}(P^T_\Delta x \otimes x) = \langle P^T_\Delta x, x \rangle$$  \hspace{1cm} (1.3)$$

In an attempt to clarify the notion of a pure state, let us observe that given a unit vector $x \in H$ for which equation (1.3) holds, we may construct a maximal abelian $C^*$-sub-algebra of operators all of which have $x$ as an eigenvector. To do this, begin first with the algebra generated by $x \otimes x$ and the identity operator. This is an abelian $C^*$-algebra, and so it is contained in some maximal abelian $C^*$-algebra, in which every member commutes with $x \otimes x$, so every member has $x$ as an eigenvector. Physicists say that observables corresponding to commuting operators are compatible, meaning they can be measured simultaneously. According to Dirac, [6], $x$ is an eigenvector of $T$ if and only if a measurement of $T$ in a state corresponding to $x \otimes x$, yields the value $\langle Tx, x \rangle$ with probability one. Consequently, there is maximal information about a physical system when it is in a pure state.

Henceforth we shall only discuss bounded operators. In this case, if $U$ is a density-operator, then for each $T \in B(H)$, $UT$ has finite trace, so the equation $\omega_U(T) = \text{tr}(UT)$ defines a norm-1 linear functional. Moreover, $\omega$ is a positive functional, and normalized, as $\omega(I) = \text{tr}(U) = 1$. Such functionals are called normal states. It can be shown ([7]) that the normal states form a convex set whose extreme points are precisely the operators $\omega_x \otimes x$, which we denote for brevity by $\omega_x$. This fact motivates the definitions that we gave prior to the physical discussion.
1.4 Pure states and representation theory

Not all pure states of a $C^*$-algebra are $\omega_x$ for some unit vector $x \in H$. Indeed, for the diagonal algebra $\mathbb{D}$, we only have a countable sequence of pure states of the form $\omega_x$, namely, $\omega_{e_n}$ for the standard unit vectors, while on the other hand we have uncountably many homomorphisms on $\mathbb{D}$ (the space of homomorphisms of $\mathbb{D}$ is $\beta\mathbb{N}$ - Stone-Cech compactification of $\mathbb{N}$; see the next chapter). Pure states not of the form $\omega_x$ are called singular states. A pure state is singular if and only if it annihilates all compact operators. One direction is clear -if a pure state is not singular then it is $\omega_x$ for some unit vector and it of course does not vanish on the compact operator $x \otimes x$. The proof of the other direction, that a singular pure state vanishes on all compact operators, requires some tools from representation theory of $C^*$-algebras. See, e.g., [13], vol. II.

For an abelian $C^*$-algebra, the set of homomorphisms coincides with the pure states. As we shall presently see, every pure state can be extended to a pure state of $B(H)$. Therefore, an alternative formulation to Problem 1.2.1, which is also the original formulation of the problem, is the following:

**Problem 1.4.1.** Let $A$ be a maximal abelian $C^*$ sub-algebra of $B(H)$. Can every pure state of $A$ be uniquely extended to a pure state of $B(H)$?

The answer, as was already mentioned, is in general negative., but for $A = \mathbb{D}, H = l^2(\mathbb{N})$, is still open. In this case the problem has implications on the central problem of $C^*$-algebra theory, namely the problem of classifying irreducible representations of $B(l^2(\mathbb{N}))$. A positive answer to the KS problem for $A = \mathbb{D}$ will characterize, up to equivalence, a class of irreducible representations of $B(l^2(\mathbb{N}))$, as follows: a theorem by Gelfand-Neymark states that given a positive functional $f$ on $B(l^2(\mathbb{N}))$, there is a cyclic representation $T \to \pi_T$ of $B(l^2(\mathbb{N}))$ in a Hilbert space $H'$ such that

$$f(T) = \langle \pi_T \xi, \xi \rangle, \xi \in H', \|\xi\| = 1, T \in B(l^2(\mathbb{N}))$$ (1.4)

If in (1.3) $f$ is a state of $B(l^2(\mathbb{N}))$ then the corresponding representation is irreducible if and only if $f$ is a pure state. ([13], Vol. II, Th. 10.23). Let us examine under what conditions the restriction of a pure state $f$ to the diagonal algebra $\mathbb{D}$ is a pure state of $\mathbb{D}$. If the restriction $f_\mathbb{D}$ is a pure state, then on $\mathbb{D}$ $f$ is multiplicative so for all $D \in \mathbb{D}$:

$$|f(D)| = |\langle \pi_D \xi, \xi \rangle| \leq \|\pi_D\xi\|$$

$$= \langle \pi_D \xi, \pi_D \xi \rangle^{1/2} = \langle \pi_{D^*D} \xi, \xi \rangle^{1/2}$$

$$= f(D^*D)^{1/2} = (|f(D)|^2)^{1/2} = |f(D)|$$

Since we have equality in Schwarz' inequality, we infer that $\xi$ is an eigenvector of all operators $\pi_D, D \in \mathbb{D}$, with corresponding eigenvalue $f(D)$. In other
words, the restriction of the representation $\pi$ to $D$ gives a one-dimensional representation of $D$ in the space spanned by $\xi$. On the other hand it is clear that if $\xi$ is an eigenvector of all the operators $\pi D$ then the functional $f(D) = \langle \pi D \xi, \xi \rangle$ is multiplicative, hence pure, as a state of $D$. (Along these lines we can show that if $A$ is an abelian $C^*$-algebra of operators on a Hilbert space $H$, then the state $\omega_x = \langle Tx, x \rangle$ is a pure state of $A$ if and only if $x$ is an eigenvector of all the operators in $A$. ) We have proved that whenever $f$ is a state of $B(l^2(\mathbb{N}))$, the restriction of $f$ to $D$ is a pure state if and only if the restriction of the representation given by (1.4) is one-dimensional.

Subsequently, an affirmative answer to the KS problem will yield that the representation in (1.4) is irreducible if and only if it’s restriction to $D$ is one-dimensional. These considerations show that the problem of classification of irreducible representations of $B(l^2(\mathbb{N}))$ is more general than the KS problem. A solution of the former will solve the latter, but even if the KS problem were to be solved affirmatively, there are not enough pure states of $D$ in order to characterize all the irreducible representations of $B(l^2(\mathbb{N}))$ by considering their restrictions to $D$. Indeed, not all pure states of $B(l^2(\mathbb{N}))$ are extensions of pure states of $D$. To see this, it suffices to take a state of the form $\omega_x$ where $x$ is a unit vector which is not a standard basis vector.

$x$ is not an eigenvector of all the diagonal operators, and so the restriction of $\omega_x$ to $D$ cannot be a pure state. Nevertheless, $\omega_x$ is a pure state of the bigger algebra $B(l^2(\mathbb{N}))$. A short way to see this is to first prove:

**Theorem 1.4.1.** Let $A$ be a maximal abelian $C^*$ sub-algebra of $B(H)$. If the state $\omega_x(T) = \langle Tx, x \rangle$ is a pure state of $A$, then it has a unique extension to a pure state of $B(H)$.

**Proof.** This proof is taken from [14]. The projection $x \otimes x$ commutes with every operator for which $x$ is an eigenvector. Therefore $x \otimes x$ commutes with all operators of $A$, and since $A$ is maximal, $x \otimes x$ must belong to $A$. If $\omega$ is any extension of $\omega_x$ to a state of $B(H)$, then $\omega$ is not singular because it does not vanish on the compact $x \otimes x$. Therefore there is a unit vector $y \in H$ such that $\omega = \omega_y$. On $A \omega_y = \omega_x$ so

$$1 = \omega_x(x \otimes x) = \omega_y(x \otimes x) = \langle \langle y, x \rangle x, y \rangle = |\langle x, y \rangle|^2$$

(1.5)

So $y$ belongs to the range of $x \otimes x$, which means that $y = tx$ for some $t$, and since $\|x\| = \|y\| = 1$, we must have $|t| = 1$. As a result, for every $T \in B(H)$:

$$\omega_y(T) = \langle Ty, y \rangle = |t|^2 \langle Tx, x \rangle = \omega_x(T)$$

(1.6)

which shows that $\omega_x$ is the unique extension of its restriction to $A$. 

Armed with this result, in order to show that $\omega_x$ is a pure state of $B(l^2(\mathbb{N}))$ we construct, as before, a maximal abelian $C^*$ sub-algebra $A$ of
operators all having $x$ as an eigenvector. The restriction of $\omega_x$ to $A$ is a pure state, and so by Theorem 1.4.1 $\omega_x$ itself is a pure state.

Every pure state of $A$ has at least one extension to a pure state of $B(l^2(\mathbb{N}))$. Theorem 1.1 shows that the states $\omega_{e_n}$ on $\mathbb{D}$ have a unique extension to pure states of $B(l^2(\mathbb{N}))$. We shall see in the next chapter how this result can be obtained in a different way.
Chapter 2

The pure states of $\mathbb{D}$

2.1 Preliminaries

A $C^*$-algebra is a Banach-algebra $A$ over the complex field $\mathbb{C}$, equipped with a unit $e$ and an involution operation $x \rightarrow x^*$ such that $\|x^*x\| = \|x\|^2$. There are two main types of $C^*$ algebras: the abelian ones, modeled as an algebra $C(K)$ of continuous complex-valued functions on some compact Hausdorff space $K$, with the maximum norm and an involution operation given by conjugation: $f \rightarrow \overline{f}$, and the non-abelian ones, modeled as the algebra $B(H)$ of bounded linear operators on some Hilbert space $H$, with the usual norm and involution furnished by the adjoint operation $T \rightarrow T^*$. Every $C^*$ algebra $A$ is isomorphic to a closed sub-algebra of $B(H)$ for a suitable Hilbert space $H$, and the isomorphism preserves involution, that is, if $x \in A$ is mapped to $\phi x$, then $x^*$ is mapped to $(\phi x)^*$. ([15], Th. 12.41). Abelian $C^*$ algebras are $*$-isomorphic and isometric to $C(\Delta)$, where $\Delta$ is the space of homomorphisms of $A$. A homomorphism of $A$ is a linear multiplicative functional on $A$, and the topology on $\Delta$ is that induced by the $w^*$-topology of $A'$, the dual space of $A$.

A linear functional $f$ on a $C^*$ algebra $A$ is called positive if $f(x^*x) \geq 0$ for every $x \in A$. A convenient criterion for checking positivity of a linear functional $f$ is this: $f$ is positive if and only if $f$ is bounded and $\|f\| = f(e)$, ([13] Th. 4.3.2.). For example, if $H$ is a Hilbert space and $y \in H$ then $f(T) = \langle Ty, y \rangle$ is a positive functional on $B(H)$. If $\|y\| = 1$ then $\|f\| = 1$. Denote by $S(A)$ the set of positive linear functionals on a $C^*$ algebra $A$, satisfying $\|f\| = f(e) = 1$. These are the states of $A$.

Proposition 2.1.1. Let $A$ be a $C^*$ algebra. The set of normalized positive functionals $S(A)$ is convex and $w^*$ compact in $A'$. Let $P(A)$ denote the set of extreme points of $S(A)$. Then $A$ is abelian if and only if $P(A)$ consists precisely of the homomorphisms of $A$.

Proof. By the Hahn-Banach theorem, there exists a bounded linear functional $f$ such that $f(e) = |e| = 1$, so $S(A) \neq \emptyset$. If $f, g \in S(A)$ and $0 < t < 1$
then for each $x \in A$
\[
(tf + (1 - t)g)(x^*) = tf(x^*) + (1 - t)g(x^*) \geq 0
\]
and $(tf + (1 - t)g)(e) = 1$, so $S(A)$ is convex. If $F \in \overline{S(A)}^{w^*}$, then given $\varepsilon > 0$ the set
\[
W_x = \{ G \in A' : |G(e) - F(e)| < \varepsilon \} \cap \{ G \in A' : |G(x^*) - F(x^*)| < \varepsilon \}
\]
is $w^*$-open, containing $F$, and intersecting $S(A)$ for every choice of $x \in A$. Thus there exist $f, g \in S(A)$ such that $|f(e) - F(A)| < \varepsilon$ and $|g(x^*) - F(x^*)| < \varepsilon$ so $F(x^*) \geq 0$ for all $x \in A$ and $F(e) = 1$. Hence, $\overline{S(A)}^{w^*} = S(A)$.

$S(A)$ is norm-bounded and $w^*$-closed, so by the Banach-Alaoglu theorem, it is $w^*$-compact. By the Krein-Milman theorem, $P(A) \neq \emptyset$. If $A$ is abelian, $A$ is $*$-isomorphic and isometric to $C(\Delta)$ where $\Delta$ is the homomorphism-space of $A$. The equation
\[
F(x) = \int_{\Delta} \hat{x} \, d\mu
\]
defines a 1-1 correspondence between $S(A)$ onto the set of probability measures on $\Delta$, where $\hat{x}$ is the Gelfand transform of $x \in A$. ([15], Th. 11.32). Extreme points of $S(A)$ correspond to measures concentrated at points of $\Delta$ and vice versa. Thus $P(\Delta) = \Delta$. If $P(\Delta)$ is the homomorphism space of $A$, then for each $x, y \in A$ and $f \in P(A)$, $f(xy - yx) = 0$. This will imply that $xy = yx$. Without loss of generality, $A$ is a closed sub-algebra of $B(H)$ for a suitable Hilbert space $H$. Since $S(A) = \overline{\text{conv}P(A)}^{w^*}$, we have $F(xy - yx) = 0$ for each $F \in S(A)$. Let $\xi \in H$ be a unit vector. Then $\langle (xy - yx)\xi, \xi \rangle = 0$ and so $\langle xy\xi, \xi \rangle = \langle yx\xi, \xi \rangle$ for all unit vectors $\xi \in H$. Hence $xy = yx$, and $A$ is abelian. \qed

Remark 2.1.1. In general $P(A)$ is not $w^*$-closed in $A'$. Indeed, if $H$ is an infinite dimensional Hilbert space, then $P(B(H))$ is not $w^*$-closed; see [13], vol. I 4.6.69.

2.2 The pure states of the algebra $l^\infty(\mathbb{N})$

Take $A = l^\infty(\mathbb{N})$ over $\mathbb{C}$. $P(A)$ can be identified with $\beta \mathbb{N}$, the Stone-Cech compactification of $\mathbb{N}$. To see this, take $f \in P(A)$ and $x \in A$. Then $f(x)$ lies in the spectrum of $x$. (Proof: If not, then $x - f(x)e$ is invertible with some inverse $y$, and then $1 = f(e) = f((x - f(x)e)y) = f(x - f(x)e)f(y) = 0$). Now, the spectrum of $x$ is $\{x(n)\}_{n=1}^{\infty}$ and so given $\varepsilon > 0$ and $x_1, \ldots, x_k \in A$, there is an $n \in \mathbb{N}$ such that
\[
\max_{1 \leq j \leq k} |f(x_j) - x_j(n)| < \varepsilon
\]

(2.2)
2.2. THE PURE STATES OF THE ALGEBRA $L^\infty(\mathbb{N})$

Denote by $\omega_n$ the homomorphism of $A$ defined by $\omega_n(x) = x(n)$. (2.2) shows that for any $f \in P(A)$, every $w^*$-neighborhood of $f$ meets $\{\omega_n\}_{n \in \mathbb{N}}$ and so $P(A) \subset \overline{\{\omega_n\}_{n \in \mathbb{N}}}$. On the other hand, $P(A)$ is $w^*$-compact, and $\omega_n \in P(A)$ for all $n \in \mathbb{N}$, hence $P(A) \supset \overline{\{\omega_n\}_{n \in \mathbb{N}}}$, so we have inclusions in both directions and $P(A) = \overline{\{\omega_n\}_{n \in \mathbb{N}}}$. We shall make use of the identification between the pure states on $A$ and the collection of ultra-filters in $\mathbb{N}$.

**Theorem 2.2.1.** Every ultra-filter $U$ in $\mathbb{N}$ defines a homomorphism $f \in P(A)$ by $f_U(x) = \lim_U x(n)$, and given a homomorphism $f \in P(A)$ there exists a unique ultra-filter $U$ in $\mathbb{N}$ such that $f = f_U$.

**Proof.** To construct $U$, define for each subset $\sigma \subset \mathbb{N}$ a member $x_\sigma$ of $l_\infty$ by

$$x_\sigma = 1 \iff n \in \sigma$$

and define a family $U$ of subsets by:

$$U = \{\sigma \subset \mathbb{N} : f(x_\sigma) = 1\}$$

Since for each $\sigma$: $x_\sigma^2 = x_\sigma$, the fact that $f_\sigma$ is multiplicative implies $f(x_\sigma) = f(x_\sigma^2) = f(x_\sigma)^2$, and so either $f(x_\sigma) = 0$ or $f(x_\sigma) = 1$. $U$ is the desired ultra-filter: if $\sigma \in U$, and $\tau$ some set containing $\sigma$, then $x_\sigma \leq x_\tau$, and so $f(x_\sigma) \leq f(x_\tau)$, and $f(x_\tau) = 1$. If $\sigma, \tau \in U$ then $f(x_\sigma x_\tau) = f(x_\sigma)f(x_\tau) = 1$ and $x_\tau \cdot x_\sigma = x_{\tau \cap \sigma}$, so that the intersection $\tau \cap \sigma$ belongs to $U$. Clearly the empty set is not a member of $U$. These considerations already show that $U$ is a filter, and in order to show that it is actually an ultra-filter, it suffices to show that for each $\sigma \subset \mathbb{N}$, exactly one of the sets $\sigma, \sigma^c$ is contained in $U$, and this follows immediately from the equality $f(x_\sigma + x_{\sigma^c}) = f(e) = 1$.

It remains to show that $\lim_U x(n) = f(x)$ for all $x \in A$. (The limit exists because $\{x(n)\}_{n=1}^\infty$ is bounded). Recall that for an ultra-filter $U$ the limit $\lim_U x(n)$ is equal to some $y$ if and only if for every open set $O$ containing $y$, the set $\{n : x(n) \in O\}$ belongs to $U$. So we need to show that for each $\varepsilon > 0$ the set $\sigma = \{n \in \mathbb{N} : |x(n) - f(n)| < \varepsilon\}$ belongs to $U$, that is, $f(x_\sigma) = 1$. $\sigma$ is not empty, because as was remarked earlier $f(x)$ belongs to the spectrum of $x$ which is the closure of the sequence $\{x(n)\}_{n=1}^\infty$. Let $\hat{x}_\sigma$ denote the Gelfand transform of $x_\sigma$. Since $x_\sigma^2 = x_\sigma$ and $\hat{x}_\sigma$ is continuous, $\hat{x}_\sigma$ must be the characteristic function in $P(A)$ of the closed and open set $\hat{x}_\sigma^{-1}\{1\}$. If $\hat{x}_\sigma(f) = 0$, then there is a neighborhood $W \subset P(A)$ of $f$ such that for each $g \in W$, $\hat{x}_\sigma(g) = 0$. But the topology of $P(A)$ is homomorphic to the $w^*$-topology of the dual $A'$, and since $f(x) \in \{x(n)\}_{n=1}^\infty$, we deduce that every $w^*$-open set containing $f$ contains also a homomorphism of the form $\omega_n$, where $n$ may also be picked from the set $\sigma$. As a result, since $\hat{x}_\sigma(\omega_n) = 1$, we cannot have $\hat{x}_\sigma(f) = 0$, and so $\hat{x}_\sigma(f) = f(x_\sigma) = 1$. \qed
2.3 Extensions of pure states of $\mathbb{D}$

We mentioned in the introduction that the statement "all operators in $B(H)$ have the paving property" is equivalent to the statement "all pure states of $\mathbb{D}$ have a unique extension to a pure state of $B(H)$". It was shown by Kadison and Singer that any two extensions $\rho_1, \rho_2$ of a pure state of $\mathbb{D}$ to positive functionals of $B(H)$ coincide on all operators that have the paving property. It is natural to seek a characterization of those pure states of $\mathbb{D}$ that do have a unique extension to pure states of $B(H)$. This is the purpose of this section. The main argument is taken from [14]. Anderson also proved a result along these lines. ([1]). We shall need a proposition and a lemma.

**Proposition 2.3.1.** Let $A,B$ be $C^*$-algebras, $B \subset A$, and $e \in B$.

1. Every state $f \in S(B)$ has an extension to a state $f \in S(A)$.

2. Every pure state $f \in P(B)$ has an extension to a pure state $F \in S(A)$.

3. A pure state $f \in P(B)$ has a unique extension to a pure state $F \in P(A)$ if and only if $f$ has a unique extension to a state $F \in S(A)$.

4. A pure state $f \in P(B)$ has a unique extension to a pure state $F \in P(A)$ if and only if for all $y \in A$, $y = y^*$:

$$\sup\{f(x) : x = x^* \in B, x \leq y\} = \inf\{f(x) : x = x^* \in B, x \geq y\}$$

The proofs of (1)-(3) are simple, whereas (4) is the main tool to obtain necessary and sufficient conditions for the uniqueness of extension of a particular pure state. This is in fact part of Theorem 4.3.13 in [13], and the proof can be found there.

**Proof.** (1) The Hahn-Banach theorem gives an extension $F$ of $f \in S(B)$ so that $\|F\| = \|f\|$. As $F(e) = 1 = \|f\| = f(e)$, $F$ is a norm-1 positive functional, that is, a state.

(2) Let $f \in P(B)$, and denote by $E(f)$ the set of extensions of $f$ that do not increase its norm. (1) Shows that $E(f)$ is not empty; $E(f)$ is convex, and we skip the routine check. Moreover, $E(f)$ is $w^*$-closed: If $F \in E(f)^{w^*}$ then for each $x \in B$ and $\varepsilon > 0$ we can find $G \in E(f)$ such that $|G(x) - F(x)| < \varepsilon$, and so $|f(x) - F(x)| < \varepsilon$. Since $\varepsilon$ is an arbitrary positive number, we get $f(x) = F(x)$, and this holds for all $x \in B$. Consequently, $F$ extends $f$, and $E(f)$ is therefore convex and $w^*$-compact, so it has extreme points. We will show that all it's extreme points are in fact pure states of $A$, and this will conclude the proof of (2). Assume $F$ is an extreme point of $E(f)$ but not a pure state of $A$. Then there are states $g, h \in S(A)$ so that $F = g_{+} h_{+} 2$. Restrict this equality to $B$: $f|_B = g |_{\frac{1}{2}} + h |_{\frac{1}{2}}$. Since $f$ is a pure state on $B$, it does not have non-trivial convex representations there. So $g|_B = h|_B = f$, 


and so \( g, h \) are also extensions of \( f \), that is, \( g, h \in E(f) \). However, we did assume \( g \neq h \), which contradicts \( F \) being an extreme point of \( E(f) \). So all extreme points of \( E(f) \) are pure states of \( A \).

(3) If \( f \in P(B) \) has a unique extension, then \( E(f) \) has only one extreme point, and by the Krein-Milman theorem so does \( E(f) \). The other direction is also immediate.

Lemma 2.3.1. Let \( A, B \) be \( C^* \)-algebras, \( B \subset A \), and \( B \) abelian. If \( F \in S(A) \) is an extension of a homomorphism \( f \in P(B) \), then for every \( y \in A, x \in B \):

\[
F(xy) = F(yx) = F(y)f(x)
\]

Proof. There exists a cyclic representation \( \pi_F \) of \( A \) in some Hilbert space \( H \), such that for every \( y \in A \), \( \pi_F(y) \) belongs to \( H \), and the connection between \( f \) and \( \pi_f \) is given by:

\[
F(y) = \langle \pi_F(y), \xi \rangle \quad \xi \in H, |\xi| = 1
\]

(2.5)

where \( \xi \in H \) is a cyclic vector for the representation \( y \to \pi_F(y) \). For every \( x \in B \):

\[
|f(x)| = |\langle \pi_F(x)\xi, \xi \rangle| \leq \|\pi_F(x)\|\|\xi\| = \langle \pi_F(x)\xi, \pi_F(x)\xi \rangle^{1/2}
\]

\[
= \langle \pi_F(x^*x)\xi, \xi \rangle^{1/2} = f(x^*x)^{1/2} = |f(x)|
\]

Thus we have equality in Schwartz inequality, so there must be some scalar \( \lambda \in \mathbb{C} \) for which \( \pi_F(x)\xi = \lambda \xi \). Comparing with (2.5) we see that \( \lambda = F(x) = f(x) \). Therefore for every \( x \in B, y \in A \):

\[
F(xy) = \langle \pi_F(xy)\xi, \xi \rangle = \langle \pi_F(x)\pi_F(y)\xi, \xi \rangle
\]

\[
= f(x)\langle \pi_F(y), \xi \rangle = f(x)F(y)
\]

A similar calculation for \( F(yx) \) yields the same result.

For a subset \( \sigma \subset \mathbb{N} \), let \( P_\sigma \) denote the diagonal projection defined by:

\[
n \in \sigma \iff \langle P_\sigma e_n, e_n \rangle = 1
\]

Theorem 2.3.1. Let \( f \in P(\mathbb{D}) \). A necessary and sufficient condition that \( f \) has a unique extension to a state \( F \in S(B(l_2^2)) \) is that for every \( T \in B(l_2^2) \):

\[
\inf \{\|P_\sigma(T - E(T))P_\sigma\| : \sigma \in \mathcal{U}_f\} = 0
\]

where \( \mathcal{U}_f \) is the ultra-filter corresponding to \( f \), and \( E(T) \) the diagonal of \( T \).
Proof. The condition is sufficient: In the notation of proposition 2.3.1, let \( f \in E(f) \). By the lemma, for every \( \sigma \in \mathcal{U}_f \) and every \( T \in B(l_2) \):

\[
\beta_f(T) = F(P_\sigma(T - E(T))P_\sigma) = (f(P_\sigma))^2F(T - E(T)) = f(P_\sigma^2)F(T - E(T)) = F(T - E(T))
\]

so

\[
|F(T - E(T))| = |F(P_\sigma(T - E(T))P_\sigma)| \leq \|F\|\|P_\sigma(T - E(T))P_\sigma\|
\]

Therefore if \( \beta_f(T) = 0 \) then \( F(T) = F(E(T)) = fE(T) \), so that if for every \( T \in B(l_2), \beta_f(T) = 0 \) then \( F = fE \) is the only extension of \( f \).

The condition is necessary: From the assumption it follows that \( F = fE \) is the only extension of \( f \). Let \( T \in B(l_2) \) and assume at first that \( T \) is self-adjoint and it’s diagonal is zero. From proposition 2.3.1(4), we have:

\[
0 = \sup\{f(D) : D = D^*, D \leq T\} = \inf\{f(D) : D = D^*, D \geq T\}
\]

and so given \( \varepsilon > 0 \), we can find self-adjoint \( D_1, D_2 \in \mathbb{D} \), so that \( D_1 \leq T \leq D_2 \)

and

\[
-\varepsilon < f(D_1) \leq 0 \leq f(D_2) < \varepsilon \quad (2.6)
\]

and from the continuity of \( \hat{D}_1, \hat{D}_2 \) on \( P(\mathbb{D}) \) we deduce that there exists a neighborhood \( W \) of \( f \) in \( P(\mathbb{D}) \) so that for each \( g \in \overline{W} \) inequality (2.6) holds with \( g \) in place of \( f \). Since \( P(\mathbb{D}) \) is the Stone-Cech compactification of the natural numbers, \( P(\mathbb{D}) \) is totally disconnected, whence the closure of each open set is open. Therefore the boundary of \( \overline{W} \) is not empty and so the indicator function \( \chi_{\overline{W}} \) is continuous. Let \( P_{\overline{W}} \) denote the diagonal projection satisfying \( \hat{P}_{\overline{W}} = \chi_{\overline{W}} \). Put:

\[
\sigma = \{n \in \mathbb{N} : \langle P_{\overline{W}}e_n, e_n \rangle = 1\}
\]

then \( P_{\overline{W}} = P_\sigma \). Moreover, if \( \mathcal{U}_f \) is the ultra-filter corresponding to \( f \), then \( \sigma \in \mathcal{U}_f \), because

\[
f(P_\sigma) = f(P_{\overline{W}}) = \hat{P}_\sigma(f) = \hat{P}_{\overline{W}}(f) = \chi_{\overline{W}}(f) = 1
\]

Now,

\[
\varepsilon P_\sigma \geq P_\sigma D_2 = P_\sigma D_2 P_\sigma \geq P_\sigma TP_\sigma \geq P_\sigma D_1 P_\sigma = P_\sigma D_1 \geq -\varepsilon P_\sigma
\]
2.4. GLIMM’S CONJECTURE

since for self-adjoint operators $S, T$ the inequality $-T \leq S \leq T$ implies $|S| \leq |T|$ ([13], 4.2.8), we have:

$$|P_\sigma TP_\sigma| \leq \varepsilon|P_\sigma| = \varepsilon$$

(2.7)

Since for each $\varepsilon > 0$ we can find $\sigma \in \mathcal{U}_f$ such that (2.7) holds, we must have $\beta_f(T) = 0$. If $T$ is not self-adjoint, we can write $T = T_1 + T_2$ with self adjoint $T_1, T_2$, and $T - E(T)$ is an operator whose diagonal is zero. The general case follows easily.

The preceding theorem facilitates a simple proof of the next result.

**Corollary 2.3.1.** For each $n \in \mathbb{N}$, the pure state defined on $\mathbb{D}$ by $\omega_n(D) = \langle De_n, e_n \rangle$ has a unique extension to a pure state of $B(l_2)$, given by $\Omega_n(T) = \omega_n(E(T))$.

**Proof.** The corresponding ultra-filter to $\omega_n$ is the family of subsets $\sigma \subset \mathbb{N}$ containing $n$. Since $\{n\} \in \mathcal{U}_\omega$, for each $T \in B(l_2)$ we have

$$\beta_{\omega_n}(T) \leq \|P_{\{n\}}(T - E(T))P_{\{n\}}\| = \|\langle Te_n, e_n \rangle I - \langle E(T)e_n, e_n \rangle I\| = 0$$

where $I$ denotes the identity operator, and so by the preceding theorem, $\omega_n$ has a unique extension. 

At this point arises the question whether there exist pure states on $\mathbb{D}$ that correspond to points in $\beta\mathbb{N} - \mathbb{N}$ and have a unique extension to a pure state of $B(l_2)$. Reid answered this question in the affirmative ([15]). He showed that if $\mathcal{U}$ is an ultra-filter having the following property: For every partition of $\mathbb{N}$ into disjoint intervals $[1, n_1], [n_1 + 1, n_2], \ldots$, there exists a subset $\sigma \in \mathcal{U}$ so that $\sigma$ contains precisely one member of each partition-interval, then the pure state that corresponds to this ultra-filter has a unique extension to a pure state of $B(l_2)$. At this time we know of no other examples of ultra-filters corresponding to points of $\beta\mathbb{N} - \mathbb{N}$ for which the corresponding pure state has a unique extension to a pure state of $B(l_2)$.

2.4 Glimm’s conjecture

**Definition 2.4.1.** Let $A, B$ be $C^*$-algebras, $A \subset B$. We say that $A$ separates the pure states of $B$ if given $f, g \in P(B), f \neq g$, there exists some $a \in A$ such that $f(a) \neq g(a)$.

We shall not enter here into a detailed discussion of $C^*$-algebras separating pure states of $C^*$-algebras that contain them. This topic is investigated in the general theory of $C^*$-algebras, and not all problems are solved. For example, in [9], Glimm conjectures:
Conjecture 2.4.1. Let $A, B$ be $C^*$-algebras, $A \subset B$. If $A$ separates the pure states of $B$, then $B = A$.

Separation of pure states is clearly connected to the KS problem: If $A, B$ are $C^*$-algebras, $A \subset B$ and $A$ separates the pure states of $B$, then every pure state of $A$ has a unique extension to a pure state of $B$. Here we show:

Proposition 2.4.1. An abelian $C^*$ sub-algebra never separates the pure states of a larger $C^*$-algebra. More precisely: Let $A, B$ be $C^*$-algebras, $A \subset B$ and $B \neq A$. If $A$ is abelian, then $A$ does not separate the pure states of $B$.

For the proof we need two lemmas.

Lemma 2.4.1. If $A, B$ are $C^*$-algebras, $A \subset B$ and $A$ separates the pure states of $B$, then the restriction of each pure state of $B$ to $A$ is a pure state of $A$.

Proof. We use the following characterization of pure states: A state $f$ of a $C^*$-algebra is pure if and only if the representation $\pi_f$ corresponding to $f$ by the Gelfand-Neymark-Segal theorem (the GNS representation) is irreducible. ($\pi_f$ is given by $f(x) = \langle \pi_f(x)\xi, \xi \rangle$ where $\xi$ is a normalized vector in the representation space.) A proof of this characterization appears, e.g., in [13], II, 10.23.

Let $f \in P(B)$. There exists a Hilbert space $H_f$ and a cyclic representation $x \to \pi_f(x)$ so that $f(x) = \langle \pi_f(x)\xi, \xi \rangle$ where $\xi \in H_f$ is a cyclic unit vector. Citing the characterization above, $x \to \pi_f(x)$ is an irreducible representation. The restriction of $f$ to $A$ is a representation of $A$ given by:

$$f(x) = \langle \pi_f(x)\xi, \xi \rangle \quad (x \in A) \quad (2.8)$$

To see that the restriction $f|_A$ is a pure state of $A$, it suffices to show that $\pi_f$ in (2.8) is irreducible. Let $H \subset H_f$ be a closed subspace of $H_f$, invariant under the action of the operators $\{\pi_f(x) : x \in A\}$. Let $E$ be the orthogonal projection on $H$. Take $y \in H, z \in H^\perp$ with $\|y + z\| = 1$. Then for each $x \in A$

$$\langle \pi_f(x)(y + z), (y + z) \rangle = \langle \pi_f(x)y, y \rangle + \langle \pi_f(x)z, z \rangle \quad (2.9)$$

From this we see that $\pi_f(x)y \in H$ for every $x \in A$ and $H^\perp$ is also invariant under the operators $\{\pi_f(x) : x \in A\}$. Moreover, we have

$$\forall x \in A \quad \langle \pi_f(x)(y - z), (y - z) \rangle = \langle \pi_f(x)y, y \rangle + \langle \pi_f(x)z, z \rangle \quad (2.10)$$

For $x \in B$, define $f_1, f_2$ by

$$f_1(x) = \langle \pi_f(x)(y + z), (y + z) \rangle \quad (2.11)$$
Every non-zero vector in $H_f$ is a cyclic vector for the representation $x \rightarrow \pi_f(x)$, and $y \perp z$, $y+z \neq 0$, so (2.11) and (2.12) define irreducible representations of $B$, so $f_1, f_2$ are pure states of $B$. As their restrictions to $A$ coincide, the assumption that $A$ separates points in $B$ implies that $f_1 = f_2$. A result from representation theory ensures there is a $\lambda \in \mathbb{C}, |\lambda| = 1$ such that $y+z = \lambda(y-z)$. Hence, either $y = 0$ or $z = 0$. Since $y \in H, z \in H^\perp$ were arbitrary, we infer that either $H = \{0\}$ or $H = H_f$, showing that $H_f$ does not have non-trivial subspaces invariant under the operators $\{\pi_f(x) : x \in A\}$. The representation (2.8) is therefore irreducible.

As was remarked in the introduction, there exist pure states of $B(l_2)$ whose restrictions to $\mathbb{D}$ are not pure states. For example, if $x = \frac{e_1 + e_2}{\sqrt{2}}$ with $e_1, e_2$ the first two standard unit vectors, then $x$ is a unit vector which is not an eigenvector of the diagonal operator $Dx = \langle x, e_1 \rangle$ and so $\omega_x|_{\mathbb{D}}$ is not a pure state of $\mathbb{D}$. This phenomenon is not special to $\mathbb{D} \subset B(l_2)$.

**Lemma 2.4.2.** Let $A$ be a maximal abelian $C^*$ sub-algebra of a $C^*$-algebra $B$, and $A \neq B$. There exists a pure state of $B$ whose restriction to $A$ is not pure.

**Proof.** If not, then for every pure state $f \in P(B)$ the restriction $f|_A$ is a pure state of $A$. From lemma 2.3.1 we deduce that for each $x \in B, y \in A$ and every pure state $f \in P(B), f(xy) = f(yx)$, so $yx = xy$, and from the maximality of $A, A = B$, contrary to the assumption. 

**Proof of proposition 2.4.1**

If $B$ is not abelian, then by the preceding lemmas we may replace $A$ by maximal abelian algebra $A'$ such that $A \subset A' \subset B$, and since $A'$ does not separate points in $P(B)$, neither does $A$. If on the other hand $B$ is abelian, then since $B$ can be identified with $C(P(B))$ and $A$ with $C(P(A))$, the fact that $A \neq B$ implies that the inclusion $C(P(A)) \subset C(P(B))$ is also strict. By Stone-Weierstrass, $C(P(A))$ does not separate the points of $P(B)$. Consequently, there exist $f_1, f_2 \in P(B)$ with $f_1 \neq f_2$ so that $f_1(\hat{g}) = f_2(\hat{g})$ for each $\hat{g} \in C(P(A))$, where $\hat{g}$ is the Gelfand transform of $g \in P(A)$. As a result, $f_1(g) = f_2(g)$ for every $g \in P(A)$, and $A$ does not separate the pure states of $B$.

**Example 2.4.1.** Proposition 2.4.1 shows that whenever an abelian $C^*$ algebra is injected into a larger $C^*$-algebra $B$, then the algebraic property of being abelian implies that $A$ is not large enough to separate all the pure states of $B$. As in other cases in the theory of $C^*$-algebras, the algebraic property has a topological consequence. To illustrate this, take $B = B(l_2)$ and $A = \mathbb{D}$. Let $E : B(l_2) \rightarrow \mathbb{D}$ denote the projection carrying an operator $T$ to it’s diagonal $E(T)$. For every $f \in P(B), fE$ is a state of $B(l_2)$. This fact
follows from properties of the projection $E$: $E$ is order preserving in the sense that if $T \leq S$ then $E(T) \leq E(S)$ and $|E| = 1$. Consider the map $\Lambda : P(\mathbb{D}) \to B(l^2)$ given by $\Lambda f = fE$. $\Lambda$ maps the compact space $P(\mathbb{D})$ onto a compact subspace (in the $w^*$-topology) of $S(B(l^2))$ - the state space of $B(l^2)$, where the topology of $P(\mathbb{D})$ is the Gelfand topology, i.e., the $w^*$ topology of $P(\mathbb{D})$ as a subset of $\mathbb{D}'$ - the dual space of $\mathbb{D}$. To see this, take $f \in P(\mathbb{D})$ and a $w^*$-open set $U$ containing $\Lambda f$, say, of the form

$$U = \{F \in (B(l^2))' : |F(T_i) - \Lambda f(T_i)| < \varepsilon \} \ i = 1, 2, \ldots, n$$

where $T_i$ are in $B(l^2)$. If we take

$$V = \{h \in P(\mathbb{D}) : |h(D(T_i)) - \Lambda f(T_i)| < \varepsilon \} \ i = 1, 2, \ldots, n$$

then $V$ is an open set containing $f \in P(\mathbb{D})$ and if $h \in V$ then

$$|\Lambda h(T_i) - \Lambda f(T_i)| = |h[E(T_i)] - \Lambda f(T_i)|$$

whence $AV \subset U$, so $\Lambda$ is continuous. If $\Lambda(f_1) = \Lambda(f_2)$ for $f_1, f_2 \in P(\mathbb{D})$ then for every $T \in B(l^2)$ $f_1(E(T)) = f_2(E(T))$ so $f_1, f_2$ coincide on $\mathbb{D}$, whence $f_1 = f_2$. This shows that $\Lambda : P(\mathbb{D}) \to S(B(l^2))$ is a continuous injection of the pure states of $\mathbb{D}$ into the states of $B(l^2)$. Now, the set of states of a separable Hilbert space is never $w^*$-closed ([13]), and certainly is not $w^*$-compact. Therefore, even if the KS conjecture holds, and every pure state of $\mathbb{D}$ does indeed have a unique extension to a pure state of $B(l^2)$, in which case the unique extension is explicitly given by a composition with the diagonal, $fE$, even in that case there will always be pure states of $B(l^2)$ that cannot be obtained as extensions of pure states of $\mathbb{D}$ - and this is due to topological reasons: $P(B(l^2))$ is not $w^*$ compact, but $P(\mathbb{D})$ as well as it’s embedding into $S(B(l^2))$ are indeed compact.

To end this chapter, let us remark that there is always an order-preserving projection $\Theta : B \to A$ when $A$ is maximal abelian in a $C^*$-algebra $B$. This was proved by Von-Neumann. Therefore we can always embed $P(A)$ inside $S(B)$, as we did for $\mathbb{D} \subset B(l^2)$, so that the topological phenomenon described above is general.
Chapter 3

The Laurent Operator

3.1 Basics

For each \( \varphi \in L^\infty(0, 2\pi) \), \( L_\varphi \) is an operator from \( L^2(0, 2\pi) \) to itself, defined for each \( f \) by \( L_\varphi f = f \varphi \). The matrix representing \( L_\varphi \) with respect to the characters \( e_n(t) = e^{int}, 0 \leq t \leq 2\pi \) is given by:

\[
L_\varphi(m, n) = \langle L_\varphi e_n, e_m \rangle = \frac{1}{2\pi} \int_0^{2\pi} \varphi(t) e^{-i(m-n)t} dt = \hat{\varphi}(m-n) \quad (3.1)
\]

It is well known that the map \( \varphi \rightarrow L_\varphi \) is a \(*\)-isometric and isomorphic embedding of the \( C^* \) algebra \( L^\infty(0, 2\pi) \) into \( B(L^2(0, 2\pi)) \), so that the norm of the Laurent operator \( L_\varphi \) is just \( ||\varphi||_{L^\infty} \). In fact, \( L_\varphi \) is a normal operator, and so it’s norm equals it’s spectral radius ([15] Ex.20, p. 325), which is just \( ||\varphi||_{L^\infty} \). The operator-algebra \( L = \{ L_\varphi : \varphi \in L^\infty(0, 2\pi) \} \) is a maximal abelian \( C^* \) sub-algebra of \( B(L^2(0, 2\pi)) \). (Maximality is not trivial. For a proof, see [11] problem 115).

Kadison and Singer showed in [14] that there are pure states of \( L \) that do not have a unique pure state extension to \( B(L^2(0, 2\pi)) \). In this context, \( L \) is the algebra whose pure states are investigated. On the other hand, when we investigate the paving property of Laurent operators, it is rather the diagonal sub-algebra of \( B(l^2(\mathbb{Z})) \) whose pure states we are looking at. Every time we are looking at the paving property of a single Laurent operator, or an algebra of such operators, we should keep in mind that we are talking about uniqueness of extension of pure states of \( \mathbb{D} \), the diagonal sub-algebra of \( B(l^2(\mathbb{N})) \) (or \( B(l^2(\mathbb{Z})) \)). We note in passing that whereas \( B(l^2(\mathbb{Z})) \) is the natural habitat of the Laurent operators’ matrix representations, the space \( B(l^2(\mathbb{N})) \) naturally accommodates the Toeplitz operators \( T_\varphi \), obtained from the Laurent operators by applying the orthogonal projection onto the space of functions whose Fourier coefficients that correspond to negative integers all vanish.
In the first chapter we introduced, for each $T \in B(l^2(N))$, the number 
\[ \beta_f(T) = \inf\{\|P_\sigma(T - E(T))P_\sigma\| : \sigma \in \mathcal{U}_f\} \]
where $f \in P(\mathbb{D})$, and $\mathcal{U}_f$ is the corresponding ultra-filter. We may replace $N$ by $\mathbb{Z}$, and then $\mathcal{U}_f$ will be an ultra-filter over $\mathbb{Z}$. The next proposition follows immediately from the definitions:

**Proposition 3.1.1.** For every $S, T \in B(l^2(\mathbb{Z}))$, $f \in P(D)$, $\lambda \in \mathbb{C}$ and a sequence $T_n \in B(l^2(\mathbb{Z}))$ such that $T_n \to T$ in the norm-topology, one has

1. $\beta_f(T + S) \leq \beta_f(T) + \beta_f(S)$
2. $\beta_f(\lambda T) = |\lambda|\beta_f(T)$
3. $\lim_{n \to \infty} \beta_f(T_n) = \beta_f(T)$

Together with theorem 2.3.1 this proposition shows that the collection of operators $T \in B(l^2(\mathbb{Z}))$ that have the paving property, forms a Banach space. This comment justifies the fact that in the course of this chapter we shall restrict our attention to Laurent operators $L_\varphi$ whose $\varphi$ is a real valued function.

### 3.2 The family $H(\varphi, \varepsilon)$

Let $Y = \{0, 1\}^\mathbb{Z}$. Here is a metric that makes $Y$ compact:

\[ d(x, y) = \inf\left\{\frac{1}{k + 1} : x(n) = y(n) \mid n \leq k\right\} \quad (3.2) \]

Indeed, this metric induces the Tychonoff topology on $Y$. The longer the interval on which $x, y$ agree, the shorter is the distance between them. Let $T$ denote the translation operator $Tx(n) = x(n + 1)$. $T$ is continuous with respect to our metric, because if $x, y$ agree on $[-N, N]$, then $Tx, Ty$ agree at least on $[-N + 1, -N - 1]$.

We defined in the introduction the following family of subsets of $\mathbb{Z}$:

\[ H(\varphi, \varepsilon) = \{\Lambda \subset \mathbb{Z} : \|P_\Lambda(L_\varphi - E(L_\varphi))P_\Lambda\| \leq \varepsilon\} \quad (3.3) \]

Our first task is to identify $H(\varphi, \varepsilon)$ with a compact subset of $Y$, where we identify subsets $\Lambda \subset \mathbb{Z}$ with $x \in Y$ such that $x(n) = 1$ if and only if $n \in \Lambda$. We start with a simple lemma.

**Lemma 3.2.1.** If $x_n$ is a sequence in a Hilbert space $H$ converging in norm to $y$, then for every bounded operator $T \in B(H)$, $\langle Tx_n, x_n \rangle \to \langle Ty, y \rangle$ as $n \to \infty$. 
3.2. THE FAMILY $H(\varphi, \varepsilon)$

**Proof.**

$$|\langle Tx_n, x_n \rangle - \langle Ty, y \rangle| = |\langle T(x_n - y), x_n - y \rangle + \langle Tx_n, y \rangle + \langle Ty, x_n \rangle - 2\langle Ty, y \rangle|$$

$$\leq |\langle T(x_n - y), x_n - y \rangle| + |\langle T(x_n - y), y \rangle| + |\langle Ty, x_n - y \rangle|$$

$$\leq \|T\|\|x_n - y\|^2 + \|T\|\|x_n - y\|\|y\| + \|T\|\|y\|\|x_n - y\|$$

**Lemma 3.2.2.** Let $\varphi \in L^\infty(0, 2\pi)$ real valued, and $\varepsilon > 0$. Then:

1. For $\Lambda \subset \mathbb{Z}, k \in \mathbb{Z}$:

$$|P_\Lambda(L_\varphi - E(L_\varphi))P_\Lambda| = \|P_{\Lambda+k}(L_\varphi - E(L_\varphi))P_{\Lambda+k}\|$$

2. Given natural $N$, put $\Lambda_N = \Lambda \cap [-N, N]$. Then

$$\|P_\Lambda(L_\varphi - E(L_\varphi))P_\Lambda\| = \sup_N \|P_{\Lambda_N}(L_\varphi - E(L_\varphi))P_{\Lambda_N}\|$$

$$\leq \lim_{N \to \infty} \|P_{\Lambda_N}(L_\varphi - E(L_\varphi))P_{\Lambda_N}\|$$

3. $\Lambda \in H(\varphi, \varepsilon)$ if and only if for every finite subset $\sigma \subset \Lambda, \sigma \in H(\varphi, \varepsilon)$.

**Proof.** As $\varphi$ is real valued, $P_\Lambda(L_\varphi - E(L_\varphi))P_\Lambda$, which is equal to $P_\Lambda(L_\varphi - \hat{\varphi}(0)I)P_\Lambda$ is self-adjoint for every $\Lambda \subset \mathbb{Z}$, so:

$$|P_\Lambda(L_\varphi - E(L_\varphi))P_\Lambda\| = \sup_{|f| \leq 1} |\langle P_\Lambda(L_\varphi - \hat{\varphi}(0)I)P_\Lambda f, f \rangle|$$

$$= \sup_{|f| \leq 1, f \in \text{im} P_\Lambda} |\langle f, \varphi \rangle - \hat{\varphi}(0)\|f\|^2|$$

since for every $k, n \in \mathbb{Z}$ and $f \in L^2(0, 2\pi)$, $\hat{f}_{e^{-k}}(n) = \hat{f}(n + k)$, and since we also have the equalities:

$$f \in L^2(0, 2\pi), k \in \mathbb{Z} : \langle (fe^{-k})\varphi, fe^{-k} \rangle = \langle f\varphi, f \rangle$$

and

$$f \in L^2(0, 2\pi), k \in \mathbb{Z} : \|fe^{-k}\| = \|f\|$$

we deduce that 1. holds.

To prove 2, we note that for $N \in \mathbb{N}$ and $\Lambda \subset \mathbb{Z}$, we have $\Lambda_N \subset \Lambda_{N+1}$, hence:

$$\beta_N = \|P_{\Lambda_N}(L_\varphi - \hat{\varphi}(0)I)P_{\Lambda_N}\| \leq \|P_{\Lambda_{N+1}}(L_\varphi - \hat{\varphi}(0)I)P_{\Lambda_{N+1}}\| = \beta_{N+1}$$

so $\{\beta_N\}_{N=1}^{\infty}$ is a non-decreasing sequence of non-negative real numbers bounded by $\alpha = \|P_\Lambda(L_\varphi - \hat{\varphi}(0)I)P_\Lambda\|$. The proof of 2. will be complete.
once we show that \( \alpha = \sup_N \beta_N \). Indeed, given \( \delta > 0 \), (3.5) shows that there is some \( f_0 \in \text{Im} \mathcal{P}_\Lambda \), \( \|f_0\| \leq 1 \) such that

\[
|\langle f_0 \varphi, f_0 \rangle - \varphi(0)\|f\|^2| - \alpha | < \delta/2
\]

since \( S_N f_0 = \sum_{n \in \Lambda_N} f_0(n) e_n \) is a sequence converging in \( L^2(0,2\pi) \) to \( f_0 \), the previous lemma implies that there is \( N > 0 \) for which

\[
|\langle (S_N f_0) \varphi, S_N f_0 \rangle - \varphi(0)\|S_N f\|^2| - \alpha | < \delta/2
\]

(3.9)

from which it follows that

\[
\alpha - \delta < |\langle (S_N f_0) \varphi, S_N f_0 \rangle - \varphi(0)\|S_N f\|^2|
\]

and since \( S_N f_0 \in \text{Im} \mathcal{P}_\Lambda \) this implies:

\[
\alpha - \delta < \sup \{(g \varphi, g) - \varphi(0)\|g\|^2| \|g\| \leq 1 \ g \in \text{Im} \mathcal{P}_\Lambda \}
\]

(3.10)

and the r.h.s of this last inequality is just \( \|P_{\Lambda_N} (L \varphi - \varphi(0)J) P_{\Lambda_N} \| \). Thus \( \alpha \) is indeed the supremum of \( \{\beta_N\}_{N=1}^\infty \), and since \( \beta_N \) is non-decreasing \( \alpha \) is also the limit of \( \beta_N \) as \( N \to \infty \). The proof of 2. is complete. 3. follows immediately from 2.: every finite subset of \( \Lambda \) is \( \Lambda_N \) for all sufficiently large \( N \).

Now for each natural number \( N \), consider the set

\[
X_N = \{x \in Y : \{n : x(n) = 1, |n| \leq N\} \in H(\varphi, \varepsilon)\}
\]

\( X_N \) represents all subsets \( \Lambda \subset \mathbb{Z} \) whose intersection with \([-N,N]\) belongs to \( H(\varphi, \varepsilon) \). Clearly, \( X_N \supset X_{N+1} \). Moreover, \( X_N \) is compact in the metric \( d(x,y) \), because if \( y \in \overline{X_N} \), then there is an \( x \in X_N \) such that \( d(x,y) < \frac{1}{N+1} \).

In particular, \( x(n) = y(n) \) for all \( |n| \leq N \) so the set \( \{n : x(n) = y(n)\} \) also belongs to \( X_N \). Since \( X_N \) is closed in the metric compact space \( Y \), it is compact.

This is the first time we need to verify that \( H(\varphi, \varepsilon) \) is not empty, and neither is \( X_N \): a simple check shows that for every \( \varphi, \varepsilon > 0 \) and \( n \in \mathbb{Z} \), the singleton \( \{n\} \) belongs to \( H(\varphi, \varepsilon) \), and so \( X_N \neq \emptyset \) for all \( N \geq 1 \). The intersection \( X = \cap_{N=1}^\infty X_N \) is an intersection of a decreasing family of compact non-empty sets, and so \( X \) is compact, and it’s members represent subsets \( \Lambda \subset \mathbb{Z} \) with the property that \( \Lambda_N \in H(\varphi, \varepsilon) \) for every \( N \). From lemma 3.2.2 3, we see that \( \Lambda \in X \) if and only if \( \Lambda \in H(\varphi, \varepsilon) \), when we identify as usual a subset \( \Lambda \subset \mathbb{Z} \) with an element of \( Y \). Moreover, from lemma 3.2.2 1 we see that \( TX = X \), that is, \( X \) is translation-invariant. Denote by \( \emptyset \) the element \( x \in Y \) all of whose entries are zeros. At this point of the discussion it is not yet clear whether \( X \cong H(\varphi, \varepsilon) \) contains anything else than \( \emptyset \) or singletons \( \{n\} \). However, a theorem by Bourgain-Tzafriri ensures that there exists \( 0 < c = c(\varepsilon, \varphi) \) such that if \( k \geq \frac{1}{c} \) then we can find \( \sigma \subset [1,k] \) with \( |\sigma| \geq ck \) and \( \sigma \in H(\varphi, \varepsilon) \). This fact, together with a result by Rusza, enable us to prove the following result.
3.2. THE FAMILY $H(\varphi, \varepsilon)$

Theorem 3.2.1. Let $\varphi \in L^\infty(0, 2\pi)$ be real-valued, and $\varepsilon > 0$. There exists a set $\Lambda \subset \mathbb{Z}$ with positive density such that $\|P_\Lambda(L_\varphi - \hat{\varphi}(0)I)P_\Lambda\| < \varepsilon$.

Proof. The family $H(\varphi, \varepsilon)$ is homogeneous in the following sense:

1. if $\Lambda \in H(\varphi, \varepsilon)$ and $\tau \subset \Lambda$, then $\tau \in H(\varphi, \varepsilon)$.
2. if $\Lambda \in H(\varphi, \varepsilon)$ and $k \in \mathbb{Z}$ then $\Lambda + k \in H(\varphi, \varepsilon)$.

(1),(2) follow at once from lemma 3.2..2. According to a result by Rusza, if $H$ is a homogeneous system of finite subsets of $\mathbb{Z}$, there exists an infinite subset $\Lambda \subset \mathbb{N}$ such that:

(a) Every finite subset of $\Lambda$ is in $H$.
(b) The density of $\Lambda$ is given by: $\text{dens}\Lambda = \lim_{N \to \infty} \max \{\frac{|\tau \cap [1, N]|}{N} : \tau \in H\}$

Applying 1. and 2. above to the finite subsets of $H(\varphi, \varepsilon)$, we get from Rusza’s theorem that there exists a set $\Lambda \subset \mathbb{N}$ so that $\Lambda \in H(\varphi, \varepsilon)$ by part (a), and $\text{dens}\Lambda \geq c(\varepsilon, \varphi)$ by part (b) and Bourgain-Tzafriri’s theorem.

Corollary 3.2.1. Let $\varphi \in L^\infty(0, 2\pi)$ be real-valued, $\hat{\varphi}(0) \neq 0$. There exists $c = c(\varphi) > 0$ and a subset $\Lambda \subset \mathbb{N}$ with positive density such that

$$\left| \int_0^{2\pi} \varphi(t)|f(t)|^2 \, dt \right| \geq c \int_0^{2\pi} |f(t)|^2 \, dt$$

for every $f \in \text{Im}P_\Lambda$, $|f| \leq 1$.

Proof. Suppose on the contrary that for every $\delta > 0$ and every $\Lambda \subset \mathbb{N}$ with positive density we could find $0 \neq f \in \text{Im}P_\Lambda$ such that

$$\left| \int_0^{2\pi} \varphi(t)|f(t)|^2 \, dt \right| < \delta \int_0^{2\pi} |f(t)|^2 \, dt$$

Take $\varepsilon = \frac{|\hat{\varphi}(0)|}{2}$. From theorem 3.2.1 we have a set $\Lambda_1 \subset \mathbb{N}$ with positive density for which

$$\frac{|\hat{\varphi}(0)|}{2} > |P_{\Lambda_1}(L_\varphi - \hat{\varphi}(0)I)P_{\Lambda_1}| = \sup \{|\langle \varphi f, f \rangle - \hat{\varphi}(0)|f|^2| : f \in \text{Im}P_{\Lambda_1}, \|f\| \leq 1\}$$

On the other hand, by assumption we have $g \in \text{Im}P_{\Lambda_1}$, $\|g\| = 1$ such that

$$|\langle \varphi g, g \rangle| = \left| \int_0^{2\pi} \varphi(t)|g(t)|^2 \, dt \right| < \frac{|\hat{\varphi}(0)|}{3} \int_0^{2\pi} |g(t)|^2 \, dt = \frac{|\hat{\varphi}(0)|}{3}$$

combining these gives:

$$\frac{|\hat{\varphi}(0)|}{2} < |\langle \varphi g, g \rangle| < \frac{|\hat{\varphi}(0)|}{3}$$

a contradiction. \qed
CHAPTER 3. THE LAURENT OPERATOR

Remark 3.2.1. If \( B \subset (0, 2\pi) \) is a Borel set with positive Lebesgue measure, then in the previous corollary we may take \( \varphi = \chi_B \), the indicator function of \( B \). We have \( \hat{\chi}_B(0) = m(B) \), and obtain a slightly weaker version of Theorem 2.2 in [4]. The difference is that in their theorem the constant \( c \) is absolute, while we get a constant depending on the set \( B \).

We saw that \( H(\varphi, \varepsilon) \) contains sets of positive density, and at this point we do not know if any other infinite sets belong to \( H(\varphi, \varepsilon) \). Let’s show that there are infinite sets, (although of density zero), that belong to \( H(\varphi, \varepsilon) \). We need a lemma.

Lemma 3.2.3. Let \( \Lambda \in \mathbb{Z} \) and \( \Lambda - \Lambda \) the set of differences \( \{m-n : m, n \in \Lambda \} \). Let \( D\Lambda = \Lambda - \Lambda \setminus \{0\} \). The for every real-valued \( \varphi \in L^\infty \):

\[
\| P_\Lambda (L_\varphi - \hat{\varphi}(0) I) P_\Lambda \| \leq \| P_{D\Lambda} \varphi \|
\]

Proof. By lemma 2.1.1 it suffices to check the inequality on finite subsets of \( \Lambda \). Let \( \sigma \subset \Lambda \) be finite, then:

\[
\| P_\sigma (L_\varphi - \hat{\varphi}(0) I) P_\sigma \| = \sum \{|\langle \varphi f, f \rangle - \hat{\varphi}(0) \|f\|^2 : \|f\| \leq 1, \ f \in \text{Im} P_\sigma \}
\]

Given \( f \in \text{Im} P_\sigma, \|f\| \leq 1 \), we can write \( f = \sum_{n \in \sigma} c_n e_n, \sum_{n \in \sigma} |c_n|^2 \leq 1 \), and we have:

\[
|\langle \varphi f, f \rangle - \hat{\varphi}(0) \|f\|^2| = \left| \sum_{n,m \in \sigma} c_n \overline{c_m} \varphi(m-n) - \hat{\varphi}(0) \sum_{n \in \sigma} |c_n|^2 \right|
\]

\[
= \left| \sum_{n \neq m} c_n \overline{c_m} \varphi(m-n) \right|
\]

\[
< \left( \sum_{m,n \in \sigma} |c_n|^2 |c_m|^2 \right)^{1/2} \left( \sum_{m \neq n} |\varphi(m-n)|^2 \right)^{1/2}
\]

\[
\leq \left[ \sum_{n \in \sigma} |c_n|^2 \right]^{1/2} \cdot \| P_{D\Lambda} \|
\]

We now introduce a class of subsets of \( \mathbb{N} \) all belonging to \( H(\varphi, \varepsilon) \), for any \( \varepsilon > 0 \) and real-valued \( \varphi \in L^\infty(0, 2\pi) \).

Definition 3.2.1. We say that \( \Lambda = \{n_i\}_{i=1}^\infty \subset \mathbb{N} \) is of type \((k,l)\) if there exist natural numbers \( k, l \) so that

(i) The gap (i.e., difference) between any two consecutive members of \( \Lambda \) is bounded below by \( k \): \( n_{i+1} - n_i \geq k \) for all \( i \).
(ii) No natural number can be written as a difference of elements of \( \Lambda \) in more than \( l \) ways.

**Example 3.2.1.** For each \( k \), define a sequence by \( n_0 = 0, n_1 = k, \) and \( n_{j+1} = n_j + k + j \). That is, \( 0, k, 2k + 1, 3k + 3, 4k + 6 \) etc. This is an example of a set of type \((k, 1)\), as can be easily checked.

To obtain a set of type \((k, l)\) for \( l > 1 \), start with the previously defined sequence \( n_j \) and consider \( \{ln_j\} \). We have \( ln_{j+1} - ln_j = l(k + j) \). Between \( ln_j \) and \( ln_{j+1} \) insert the sequence

\[
ln_j < ln_j + (n_{j+1} - n_j) < ln_j + 2(n_{j+1} - n_j) < \ldots < ln_j + l(n_{j+1} - n_j) = ln_{j+1}
\]

The sequence thus obtained is of type \((k, l)\).

A sequence of type \((k, l)\) must have zero density, because by a theorem of Szemerédi, if \( \Lambda \subset \mathbb{N} \) has positive density then for every \( l \) \( \Lambda \) contains an arithmetic progression \( n_1, n_1 + d, \ldots, n_1 + (l - 1)d \), and in particular \( d \) can be represented as a difference of members of \( \Lambda \) in at least \( l \) ways.

**Proposition 3.2.1.** Let \( \varphi \in L^\infty(0, 2\pi) \) be real-valued, and \( \varepsilon > 0 \). The for every \( l \in \mathbb{N} \) there exists a natural number \( k = k(\varepsilon, \varphi, l) \) such that \( H(\varphi, \varepsilon) \) contains every set \( \Lambda \subset \mathbb{N} \) of type \((k, l)\).

**Proof.** Choose \( k = k(\varepsilon, \varphi, l) \) so that \( \left( \sum_{|j| \geq k} |\hat{\varphi}(j)|^2 \right)^{1/2} \leq \varepsilon l^{-1/2} \). Let \( \Lambda \subset \mathbb{N} \) be a set of type \((k, l)\). Let \( D \Lambda \) be as in lemma 3.2.3. Then:

\[
\left( \sum_{j \in D \Lambda} |\hat{\varphi}(j)|^2 \right)^{1/2} = \|P_{D \Lambda} \varphi\| \leq \left( l \sum_{|j| \geq k} |\hat{\varphi}(j)|^2 \right)^{1/2} < \varepsilon
\]

and the result follows from lemma 3.2.3. \(\square\)

Since every finite arithmetic progression of length \( k \) and common difference \( d \) is of type \((k, d)\), proposition 3.2.1 shows that \( H(\varphi, \varepsilon) \) contains arithmetic progressions of arbitrary length. We could also have deduced that from the fact that \( H(\varphi, \varepsilon) \) contains a set of positive density.

### 3.2.1 The paving property and syndetic sets

**Definition 3.2.2.** A subset \( \Lambda \subset \mathbb{Z} \) is called **syndetic** if \( \mathbb{Z} \) can be covered by a finite number of translations of \( \Lambda \), that is, if there exists \( l \in \mathbb{N} \) such that

\[
\mathbb{Z} = \Lambda + k_1 \cup \Lambda + k_2 \cup \cdots \cup \Lambda + k_l
\]

for some \( k_1, \ldots, k_l \in \mathbb{Z} \).
Clearly a syndetic set has bounded gaps. Syndetic subsets of \( \mathbb{N} \) are precisely those whose gaps are bounded, but in \( \mathbb{Z} \) we need to distinguish between the property of bounded gaps and being syndetic, as \( \{2n\}_{n=1}^{\infty} \subset \mathbb{Z} \) has bounded gaps, but no finite number of its translations cover \( \mathbb{Z} \).

Since \( H(\varphi, \varepsilon) \) is translation-invariant, if we can find a syndetic set \( \Lambda \subset H(\varphi, \varepsilon) \) then \( L_{\varphi} \) has the paving property, because if for some \( l \) and integers \( k_1, \ldots, k_l \) we have \( \mathbb{Z} = \bigcup_{i=1}^{l} \Lambda + k_i \) then we can pass to a disjoint covering of \( \mathbb{Z} \) by choosing appropriately subsets \( \sigma_i \subset \Lambda + k_i \), without increasing the norms \( |P_{\sigma_i}(L_{\varphi} - \hat{\varphi}(0))P_{\sigma_i}| \). As it turns out, the existence of a syndetic set in \( H(\varphi, \varepsilon) \) for every \( \varepsilon > 0 \) is also necessary for \( L_{\varphi} \) (\( \varphi \) real-valued) to have the paving property. This will follow from the following result (for a proof, see [8], 1.23).

**Theorem 3.2.2.** Given a partition of \( \mathbb{Z} \) into a finite number of disjoint sets \( \Lambda_1, \ldots, \Lambda_m \), there exists an \( l \in \mathbb{N} \) such that one of the sets, say \( \Lambda_j \), contains arbitrarily long finite sequences whose gaps are bounded by \( l \). That is, for every \( k \in \mathbb{N} \) we can find \( n_1 < n_2 < \cdots < n_k \) in \( \Lambda_j \) for which \( n_{i+1} - n_i \leq l \).

A set with the property described in the theorem is called piecewise syndetic ([8], 1.11).

**Lemma 3.2.4.** Let \( X \subset Y = \{0, 1\}^\mathbb{Z} \) a compact set. Assume there is an \( x \in X \) so that \( \{n | x(n) = 1\} \) is piecewise syndetic. Then \( X \) contains a \( y \) such that \( \{n | y(n) = 1\} \) has bounded gaps, and if \( X \) is translation-invariant, \( y \) may be chosen so that \( \{n | y(n) = 1\} \) is syndetic.

**Proof.** Under the assumption, there exists some \( l \in \mathbb{N} \) so that for each \( k \) there exists an element \( x_k \in X \) so that \( \{n | x_k(n) = 1\} \) corresponds to a subset \( n_1 < n_2 < \cdots < n_k \) contained in \( \{n | x(n) = 1\} \) for which \( n_{i+1} - n_i \leq l \) for all \( i = 1, 2, \ldots, k - 1 \). As \( X \) is compact, the sequence \( \{x_k\}_{k=1}^{\infty} \) has an accumulation point \( y \in X \). Put \( \Lambda = \{n | y(n) = 1\} \). Let \( n_1 < n_2 \) be consecutive members of \( \Lambda \). Take \( \delta = \frac{1}{\max\{|n_1|, |n_2|\} + 1} \), then for some \( k \) \( d(x_k, y) < \delta \). By the choice of \( \delta \), \( x_k(i) = y_k(i) \) for every \( n_1 < i < n_2 \). In particular \( x_k(n_2) = x_k(n_1) = 1 \) and so \( n_1, n_2 \) both belong to \( \{n | x_k(n) = 1\} \), and \( n_1 < n_2 \) are also consecutive in this set, in which the gaps are bounded by \( l \). Thus \( n_2 - n_1 \leq l \). This shows that \( \Lambda \) has gaps bounded by \( l \). Now, if \( X \) is translation-invariant, then for every \( k > 0 \) we can find \( x_k \in X \) such that \( x_k(n) = 1 \) if and only if \( |n| \leq [k/2] \) (where \( [m] \) is the integer part of \( m \)). In this way we will have obtained arbitrarily large finite sets, symmetric around zero, whose gaps are bounded by \( l \), all belonging to \( \{n | x_k(n) = 1\} \).

An accumulation point of such a collection will be syndetic. \( \square \)

So we have our main result:

**Theorem 3.2.3.** Let \( \varphi \in L^\infty(0, 2\pi) \) real-valued. The Laurent operator \( L_{\varphi} \) has the paving property if and only if for every \( \varepsilon > 0 \), there exists a syndetic
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set $\Lambda \subset \mathbb{Z}$ such that

$$\|P_{\Lambda}(L_{\varphi} - \hat{\varphi}(0)I)P_{\Lambda}\| < \epsilon$$

("syndetic" may be replaced with "piecewise syndetic")

Although every set of positive density contains arbitrarily long arithmetic progressions, the common difference of these is not controlled. Consequently (and unfortunately), not every set of positive density (which we have shown to be included in $H(\varphi, \epsilon)$) is piecewise syndetic:

**Example 3.2.2.** Consider the set

$$\Lambda = \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} [n \cdot 10^k, n \cdot 10^k + k)$$

For $N \in \mathbb{N}$,

$$|\Lambda \cap [1, N]| = \sum_{k=1}^{M(N)-1} \frac{N}{10^k} - R(N)$$

where the order of magnitude of $M(N)$ is the number of digits of $N$ in its decimal expansion, and $\frac{R(N)}{N} \to 0$ as $N \to \infty$. dividing both sides by $N$ and letting $N \to \infty$ we get that the density of $\Lambda$ is $\sum_{k=1}^{\infty} 10^{-k} = 1/9$. Consider the complement set $\Lambda^c$. It’s density is 8/9 but it is not piecewise syndetic. In fact, for each $l \in \mathbb{N}$, and every sequence $n_1 < n_2 < \cdots < n_{10^l}$ in $\Lambda^c$, we will find consecutive $n_j, n_{j+1}$ such that $n_{j+1} - n_j > l$. The reason is that if we choose $10^{2l}$ members of $\Lambda^c$ some of them must be greater than $10^{2l}$, but none of them belong to $[10^{2l}, 10^{2l} + 2l)$, whose length is $2l$. As a result, the gap between two such elements must exceed $l$.

We shall end this chapter with a short discussion about the case when $\varphi \in L^\infty(0, 2\pi)$ is continuous. It is known that in this case for every $\epsilon > 0$ there exists a uniform paving, that is, a paving with arithmetic progressions, of the operator $L_{\varphi}$. ([12]). Actually (HKW) showed that every $L_{\varphi}$ with $\varphi$ Riemann-integrable is (uniformly) paveable. We shall give here a short constructive proof for the continuous case, by showing that there exists a uniform paving for a class of operators whose closure contains all operators with continuous symbols. The fact that the set of paveable operators forms a closed subspace of the Banach space $B(l_2)$ (see [12]) plays a role here.

**Proposition 3.2.2.** Let $\varphi \in C(0, 2\pi)$, and $\epsilon$. There exists $d \in \mathbb{N}$ such that $H(\varphi, \epsilon)$ contains the arithmetic progression $\{dk\}_{k \in \mathbb{Z}}$, and so $L_{\varphi}$ has a uniform paving.

**Proof.** For rational $0 \leq p < q \leq \pi$, let $\chi_{pq}$ denote the indicator function of the interval $[2\pi p, 2\pi q]$. Then

$$\hat{\chi}_{pq}(n) = \frac{i}{2\pi n} e^{-in2\pi q} (1 - e^{in2\pi(p-q)})$$
Let $d$ be the first natural number for which $d(p - q) \in \mathbb{N}$. Then for all $k \in \mathbb{Z}$, $\widehat{\chi}_{pq}(kd) = 0$. Put $\Lambda = \{kd\}_{k \in \mathbb{Z}}$. Then the difference-set $\Lambda - \Lambda$ coincides with $\Lambda$. From lemma 3.2.3:

$$||P_\Lambda(L_{\chi_{pq}} - (q - p)2\pi I)P_\Lambda|| \leq ||P_{D\Lambda \chi_{pq}}|| = 0$$

$\Lambda$ is an arithmetic progression, so every $\chi_{pq}$ has a uniform paving. Given a continuous $\varphi \in C(0, 2\pi)$, uniform continuity of $\varphi$ ensures that for every $\varepsilon > 0$ we can find rational points $0 = p_0 < p_1 < \cdots < p_l$ so that $p_i - p_{i-1} < \delta(\varepsilon)$, and for every $2\pi p_{i-1} \leq s, t \leq 2\pi p_i$ $|\varphi(s) - \varphi(t)| < \varepsilon$. Choose points $\xi_i \in [2\pi p_{i-1}, 2\pi p_i]$. Then for all $0 \leq t \leq 2\pi$:

$$\left| \sum_{i=0}^{l} \varphi(\xi_i) \chi_{p_{i-1}p_i}(t) - \varphi(t) \right| = |\varphi(\xi_i) - \varphi(t)| < \varepsilon$$

because $t \in [2\pi p_{i-1}, p_i]$ for some $0 \leq i \leq l$. Thus, the closure in $L^\infty(0, 2\pi)$ of the finite linear combinations of the indicator functions $\chi_{pq}$ contains the continuous functions, and the proposition is proved. $\square$
Chapter 4

Equivalent Formulations

4.1 Six equivalent formulations

Theorem 4.1.1. The following conditions are equivalent.

Uniqueness of Extension. Every pure state $f \in P(D)$ has a unique extension to a pure state $F \in P(B(l_2))$.

Paving property. Every operator $T \in B(l_2)$ has the paving property: for every $\varepsilon > 0$ there is a partition of $\mathbb{N}$ into $m = m(\varepsilon, T)$ disjoint subsets $\sigma_1, \ldots, \sigma_m$ such that

$$
|P_{\sigma_j}(T - E(T))P_{\sigma_j}| < \varepsilon |T|, \quad j = 1, 2, \ldots, m
$$

Relative Dixmier property. Every operator $T \in B(l_2)$ has the relative Dixmier property, i.e., the set

$$
K(T) = \overline{\text{conv}}\{DTD^* : D \text{ diagonal unitary}\}
$$

contains the diagonal of $T$.

Local paving property. For every $\varepsilon$ there exists $m = m(\varepsilon)$ such that for every $n$ and every $T : l_2^n \to l_2^n$ there exists a partition of $\{1, 2, \ldots, n\}$ into $m$ disjoint subsets $\sigma_j$ so that

$$
|P_{\sigma_j}(T - E(T))P_{\sigma_j}| < \varepsilon |T|, \quad j = 1, 2, \ldots, m
$$

Restricted invertibility. There exists a real number $c > 0$ and $m \in \mathbb{N}$ so that for every $n$ and every $T : l_2^n \to l_2^n$ satisfying $|Te_i| = 1$ for $i$,

there is a partition of $\{1, 2, \ldots, n\}$ into $m$ disjoint sets $\sigma_j$, so that for every choice of scalars $a_i$:

$$
|\sum_{i \in \sigma_j} a_i Te_i| \geq c \left( \sum_{i \in \sigma_j} |a_i|^2 \right)^{1/2} \quad (j = 1, 2, \ldots, m)
$$
**Diagonalization property.** For every operator \( T \in B(l^2) \) and every pure state \( f \in P(\mathbb{D}) \):

\[
\beta_f(T) = \inf \{ \| P_\sigma(T - E(T))P_\sigma \| : \sigma \in \mathcal{U}_f \} = 0
\]

where \( \mathcal{U}_f \) is the ultra-filter corresponding to \( f \).

Kadison and Singer proved the equivalence between the paving property and the uniqueness of extension in [14] (lemma 5). The paving property has attracted considerable attention. Kadison and Singer proved that every operator represented by a permutation matrix (i.e. every row and every column contains exactly one 1 and the rest are zeros) has the paving property. ([14], Th. 3). Then Gregson [10] showed that operators all of whose matrix entries are non-negative have the paving property too. Berman, Halpern, Kaftal and Weiss independently obtained the same result ([2]). As was discussed in chapter 3, Halpern Kaftal and Weiss showed that the operator \( L_\phi \) that multiplies each \( f \in L^2[0, 2\pi] \) by \( \phi \in L^\infty(0, 2\pi) \) (a Laurent operator) has the paving property if \( \phi \) is Riemann integrable. Bourgain and Tzafriri showed that there exists a sub-algebra of the algebra of Laurent operators all of whose members have the paving property. The corresponding multipliers \( \phi \in L^\infty(0, 2\pi) \) are characterized by a certain decay condition on their Fourier coefficients ([5]). The question whether every Laurent operator has the paving property is still open.

Using characterization of singular states, it can be shown that every compact operator \( S \in B(l^2(\mathbb{N})) \) has the paving property: If \( S \) is compact and \( \rho_1, \rho_2 \) are two extensions of a singular state \( \rho \), then \( \rho_1, \rho_2 \) are also singular, and so \( \rho_1(S) = \rho_2(S) = 0 \). The diagonal of a compact operator \( E(S) \) is also a compact operator, because \( \lim_{n \to \infty} \langle Se_n, e_n \rangle = 0 \), and so \( \rho_1(S) = \rho_2(S) = \rho E(S) \). From here it follows that \( \rho_1, \rho_2, \rho E \) coincide on the two-sided ideal of the compact operators, and that \( S \) has the paving property follows from Kadison and Singer’s proof of the equivalence between the paving property and the uniqueness of extension.

All these problems about operators in \( B(l^2(\mathbb{N})) \) can be localized to \( l^n_2 \). If \( T : l^n_2 \to l^n_2 \) then \( T \) can be trivially "paved" by the projections \( P_i = e_i \otimes e_i \) where \( \{e_1, \ldots, e_n\} \) is the standard basis in \( l^n_2 \). Such pavings give no information as \( n \to \infty \). The question here is whether the local paving property holds. The only result in this direction was obtained by Bourgain and Tzafriri in [3]. They proved that for every \( \varepsilon > 0 \) and each \( M > 0 \) there is a constant \( c = c(M, \varepsilon) \) such that for each \( n \geq 1/c \) and every linear \( S : l^n_2 \to l^n_2 \) with zeros on the diagonal and \( |S| \leq M \), there exists a subset \( \sigma \subset \{1, \ldots, n\} \) whose size is at least \( nc \), such that \( \|P_\sigma SP_\sigma\| < \varepsilon \).

In [12] Halpern Kaftal and Weiss proved that if \( B \subset (0, 2\pi) \) is an open set such that the boundary \( \partial B \) has positive Lebesgue measure, and \( m(B) > 1/2 \), then for every arithmetic progression \( \Lambda \subset \mathbb{Z} \) one has \( \|P_\Lambda(L_{\chi_B} - m(B)I)P_\Lambda\| \geq m(B) \). (Here \( m(B) \) is the Lebesgue measure and \( \chi_B \) is the
characteristic function of $B$). Consequently, the authors suggested the possibility of a negative solution to the paving property, but Bourgain and Tzafriri came up with an example of an open set $V \subset (0, 2\pi)$ whose boundary has full measure yet the Laurent operator $L_{\chi_V}$ does have the paving property.

The theorem is proved by showing the following equivalences:

1. uniqueness of extension $\iff$ paving property.
2. paving property $\iff$ relative Dixmier property.
3. relative Dixmier property $\iff$ uniqueness of extension.
4. uniqueness of extension $\iff$ diagonalization property.
5. paving property $\iff$ local paving property.
6. local paving property $\iff$ restricted invertibility.

### 4.2 uniqueness of extension $\iff$ diagonalization property

This was proved in theorem 2.3.1.

### 4.3 uniqueness of extension $\implies$ paving property

This was proved by Kadison and Singer in [14]. If every $f \in P(\mathbb{D})$ has a unique extension to a pure state of $B(l_2)$, then a careful examination of the proof of theorem 2.3.1 shows that for each $f \in P(\mathbb{D})$ we can find a closed and open set $W \subset P(\mathbb{D})$ containing $f$ so that if $P_W$ is the projection corresponding to the indicator $\chi_W$, then $\|P_W(T - E(T))P_W\| < \varepsilon$, where $\varepsilon, T$ are apriori given. As $P(\mathbb{D})$ is compact, we can find a cover of $P(\mathbb{D})$ by $W_1, \ldots, W_m$, where $m$ depends on $\varepsilon, T$ only, such that $\|P_{W_j}(T - E(T))P_{W_j}\| < \varepsilon$ for all $1 \leq j \leq m$. Consider the sets

$$\sigma_j = \{n \in \mathbb{N} | \langle P_{W_j}e_n, e_n \rangle = 1\}$$

then $\bigcup \sigma_j = \mathbb{N}$, because $\bigcup W_j = P(\mathbb{D})$ implies $\sum_{j=1}^{m} P_{W_j} = I$. Passing to a disjoint covering of $\mathbb{N}$ we obtain the paving property.

### 4.4 paving property $\implies$ relative Dixmier property

Let $T \in B(l_2)$ and $\varepsilon > 0$. If $\sigma \subset \mathbb{N}$ and $\sigma^c$ is the complement of $\sigma$, then $U = P_{\sigma} - P_{\sigma^c}$ is a diagonal unitary operator, and:

$$\frac{1}{2} [T + UTU^*] = P_{\sigma}TP_{\sigma^c} + P_{\sigma^c}TP_{\sigma}$$

(4.1)
Thus, if for \( \varepsilon, T \) there are \( m = m(\varepsilon, T) \) disjoint subsets \( \sigma_1, \ldots, \sigma_m \) for which 
\[ |P_{\sigma_j}[T - E(T)]P_{\sigma_j}| < \varepsilon|T|, \] 
then from (4.1) we can write:
\[
\sum_{j=1}^{m} P_{\sigma_j}[T - E(T)]P_{\sigma_j} = \sum_{j=1}^{m} \alpha_j U_j[T - E(T)]U_j^* \tag{4.2}
\]
where \( \sum_{j=1}^{m} \alpha_j = 1, \alpha_j \geq 0 \) and \( U_j \) is unitary and diagonal, for each \( 1 \leq j \leq m \). From (4.2) we have:
\[
\| \sum_{j=1}^{m} P_{\sigma_j}(T-E(T))P_{\sigma_j} \| = \max_{1 \leq j \leq m} \| P_{\sigma_j}(T-E(T))P_{\sigma_j} \| = \| \sum_{j=1}^{m} \alpha_j U_j(T-E(T))U_j^* \| \tag{4.3}
\]
and (4.3) shows that for apriori given \( \varepsilon, T \), we have found an element in the set
\[
\text{conv} \{ U[T - E(T)]U^* | U \text{ unitary and diagonal} \}
\]
whose norm is at most \( \varepsilon|T| \). As \( \varepsilon > 0 \) is arbitrary, this means that the closure of this convex hull contains the zero operator, which equivalent to \( K(T) \) containing the diagonal of \( T \). Thus the paving property implies the relative Dixmier property. To show that \textbf{relative Dixmier property} \( \Rightarrow \text{uniqueness of extension} \) we use proposition 2.3.1. If \( f \in \mathcal{P}(\mathbb{D}) \) and \( F \) is an extension of \( f \), then for unitary and diagonal \( U \):
\[
F(U(T - E(T))U^*) = F(U)F(U^*)F(T - E(T)) = F(T - E(T))
\]
so if \( S \in K(T - E(T)) \) then \( F(S) = F(T - E(T)) \). Since the relative dixmier property implies that \( 0 \in K(T - E(T)) \), we have: \( F(T - E(T)) = F(T) - F(E(T)) = 0 \), hence \( F(T) = F(E(T)) \). In other words, the composition \( F = f \circ E \) is the only extension of \( f \) to a state of \( B(l_2) \). From proposition (2.3.1)(3) we deduce that \( f \) is a unique extension to a pure state of \( B(l_2) \).

### 4.5 paving property \( \iff \) local paving property

**paving property \( \Rightarrow \) local paving property** Take \( \tau_j = \sigma_j \cap [1, n] \).

**local paving property \( \Rightarrow \) paving property** In [2] the authors define for each \( T \in B(l^2(\mathbb{N})) \) the numbers
\[
\alpha(T) = \inf \{ \| \sum P_m T P_m \| : \sum P_m = I, \ P_i \perp P_j \ (i \neq j) \}
\]
where the infimum is taken over finite sums of mutually orthogonal projections whose sum is the identity. Similarly they denote \( \alpha_k(T) \) to be the same infimum taken over sums of \( k \) projections. They prove (prop. 2.6) that for every \( T \) and \( k \): \( \alpha_k(T) = \lim_{n \to \infty} \alpha_k((P_{[1,n]}T P_{[1,n]})) \), where \( P_{[1,n]} \) is the orthogonal projection onto the first \( n \) unit vectors. Now let \( T \in B(l^2(\mathbb{N})), \varepsilon > 0 \) be
given, with \( \|T\| \leq 1 \). Let \( m = m(\varepsilon) \) as in the local paving property. Let \( n \) be sufficiently large so that \( |\alpha_m(T - E(T)) - \alpha_m(P_{[1,n]}(T - E(T))P_{[1,n]})| < \varepsilon \), and since in the expression \( P_{[1,n]}(T - E(T))P_{[1,n]} \) we may replace \((T - E(T))\) by its restriction to \( l_2^n \), then by the local paving property we deduce that

\[
\alpha_m(P_{[1,n]}(T - E(T))P_{[1,n]}) < \varepsilon \|T\|_2 \leq \varepsilon \|T\| \leq \varepsilon
\]

and so \( \alpha_m(T - E(T)) < 2\varepsilon \).

### 4.6 Local paving property \( \implies \) restricted invertibility

Assume that restricted invertibility does not hold. Then for every \( c > 0 \) there is some \( n = n_c \in \mathbb{N} \) and a linear map \( T : l_2^n \to l_2^n \) with \( \|Te_i\| = 1 \) for all \( i \), such that whenever \( \sigma_1, \ldots, \sigma_m \) is a partition of \( [1, n_c] \) we can find some \( 1 \leq j \leq m \) and a vector \( x_0 = \sum_{i \in \sigma_j} a_i e_i \) such that

\[
\|Tx_0\| = \left\| \sum_{i \in \sigma_j} a_i Te_i \right\| < c \left( \sum_{i \in \sigma_j} |a_i|^2 \right)^{1/2} = c \|x_0\| \quad (4.4)
\]

from which it follows that:

\[
\langle T^*Tx_0, x_0 \rangle < c^2 \|x_0\|^2 \quad (4.5)
\]

With no loss of generality, we may assume \( \|x_0\| \leq 1 \). The assumption \( \|Te_i\| = 1 \) implies \( \langle T^*Te_i, e_i \rangle = 1 \) for each \( i \), and so the diagonal of \( T^*T \) is the identity matrix. Choose \( c \) so that \( c < \sqrt{\frac{2}{3}} \). Take \( \varepsilon = c^2/2 \) and use the local paving property for the operator \( T^*T \) and \( n = n_c \). There is some partition \( \tau_1, \ldots, \tau_l \) of \( \{1, \ldots, n\} \) so that

\[
\|P_{\tau_j}(T^*T - I)P_{\tau_j}\| < \varepsilon \quad \forall 1 \leq j \leq l \quad (4.6)
\]

Since \( T^*T \) is self-adjoint, so is \( P_{\tau_j}(T^*T - I)P_{\tau_j} \) for each \( j \), and so the norm can be computed as:

\[
\|P_{\tau_j}(T^*T - I)P_{\tau_j}\| = \sup \{ ||(P_{\tau_j}(T^*T - I)P_{\tau_j}x, x)|| : x \in l_2^n, \|x\| \leq 1 \} \\
= \sup \{ ||(T^*Ty, y)|| : y \in P_{\tau_j}l_2^n, \|y\| \leq 1 \}
\]

by our contradiction-assertion and the last equation:

\[
\langle T^*Tx_0, x_0 \rangle < c^2 \|x_0\|^2 \quad (4.7)
\]

\[
|x_0|^2 - c^2/2 < \langle T^*Tx_0, x_0 \rangle < |x_0|^2 + c^2/2 \quad (4.8)
\]

Combining (4.7) and (4.8) we get

\[
0 < |x_0|^2(c^2 - 1) + c^2/2 = 3c^2/2 - 1
\]

contradicting the choice of \( c \). Thus the local paving property implies restricted invertibility.
4.7 restricted invertibility \(\Rightarrow\) local paving property

We shall need the following result, proved in [3]:

**Theorem 4.7.1.** Let \(\{a_{ij}\}_{1 \leq i,j \leq n}\) be a matrix satisfying:

(i) \(a_{ij} \geq 0\) for all \(1 \leq i, j \leq n\).

(ii) \(\sum_{j=1}^{n} a_{ij} \leq 1\) for all \(1 \leq i \leq n\).

Then for every \(k\) there exists a partition \(\sigma_1, \ldots, \sigma_k\) of \(\{1, 2, \ldots, n\}\) into \(k\) disjoint subsets so that for each \(1 \leq l \leq k\):

\[
\sum_{j \in \sigma_l} a_{ij} \leq \frac{2}{k} \quad i \in \sigma_l
\]

Let \(\varepsilon > 0\) and \(S : l_2^n \to l_2^n\) a linear map with diagonal zero. Let \(s_{ij}\) be the matrix representing \(S\) with respect to the standard basis. Then:

\[
(1 \leq i \leq n) \quad \sum_{j=1}^{n} |s_{ij}|^2 = \|Se_i\|^2 \leq \|S\|^2
\]

We shall apply theorem 4.0.5 to the matrix \(a_{ij} = \frac{|s_{ij}|^2}{\|S\|^2}\). Take an integer \(k > 2\) such that \(\frac{2\|S\|^2}{k} < \varepsilon^2\), and take a partition \(\sigma_1, \ldots, \sigma_k\) of \(\{1, \ldots, n\}\) such that

\[
(i \in \sigma_k, 1 \leq l \leq k) \quad \sum_{j \in \sigma_l} a_{ij} \leq \frac{2}{k}
\]

That is:

\[
(i \in \sigma_k, 1 \leq l \leq k) \quad \|P_{\sigma_l}Se_i\|^2 = \sum_{j \in \sigma_l} |s_{ij}|^2 \leq \frac{2\|S\|^2}{k} \quad (4.9)
\]

For \(1 \leq i \leq n\) and \(1 \leq l \leq k\) put

\[
x_i^l = \frac{1}{\|S\|} \sqrt{\frac{k}{2}} P_{\sigma_l}Se_i \quad (4.10)
\]

Then we have:

(a) \(\|x_i^l\| \leq 1\) for each \(i \in \sigma_l\) and for each \(1 \leq l \leq k\).

(b) For every choice of \(\{c_i\}_{i \in \sigma_l}\):

\[
\|\sum_{i \in \sigma_l} c_i x_i^l\| \leq \sqrt{\frac{k}{2}} \left(\sum_{i \in \sigma_l} |c_i|^2\right)^{1/2} \quad (4.11)
\]
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(a) follows from 4.9 and 4.10, and (b) follows because for every $u = \sum_{i=1}^{n} c_i e_i$:

$$\|P_{\sigma_l} S P_{\sigma_l} u\| \leq \|P_{\sigma_l} S\| \|P_{\sigma_l} u\| \leq \|S\| \left(\sum_{i \in \sigma_l} |c_i|^2\right)^{1/2}$$

$$\implies \|P_{\sigma_l} S \sum_{i \in \sigma_l} c_i e_i\| \leq \|S\| \left(\sum_{i \in \sigma_l} |c_i|^2\right)^{1/2}$$

$$\implies \left\| \sum_{i \in \sigma_l} c_i P_{\sigma_l} e_i \right\| \leq \|S\| \left(\sum_{i \in \sigma_l} |c_i|^2\right)^{1/2}$$

$$\implies \left\| \sum_{i \in \sigma_l} c_i \|S\| \sqrt{\frac{2}{k}} x_{i}^l \right\| \leq \|S\| \left(\sum_{i \in \sigma_l} |c_i|^2\right)^{1/2}$$

and this is equivalent to (b).

The idea is now to partition the sets $\sigma_1, \ldots, \sigma_l$ recursively, each time reducing the number $\sqrt{k/2}$ appearing in (b). In the first step, we define $k$ new matrices $B_1, \ldots, B_k$ by:

$$(1 \leq l \leq k), \quad (M = \sqrt{k/2}) : \quad (b_{ij})^l = \frac{M^2}{M^2 - 1} \delta_{ij} - \frac{\langle x_{i}^l, x_{j}^l \rangle}{M^2 - 1}, \quad i, j \in \sigma_l$$

The given $u$ in the range of the projection $P_{\sigma_l}$, we have $\langle B_l u, u \rangle \geq 0$, for each $1 \leq l \leq k$. Indeed, if $u = \sum_{i \in \sigma_l} c_i e_i$, then

$$(1 \leq l \leq k) \quad \langle B_l u, u \rangle = \sum_{i, j \in \sigma_l} (b_{ij})^l c_i c_{\bar{j}}$$

$$= \frac{1}{M^2 - 1} \sum_{i, j \in \sigma_l} [M^2 \delta_{ij} - \langle x_{i}^l, x_{j}^l \rangle] c_i c_{\bar{j}}$$

$$= \frac{M^2}{M^2 - 1} \sum_{i \in \sigma_l} |c_i|^2 - \frac{\|\sum_{i \in \sigma_l} c_i x_{i}^l\|}{M^2 - 1} \geq 0$$

the last inequality being true thanks to (b). Since the $B_l$’s are positive operators, they have square roots, i.e., $S_l^2 = B_l$. Put $z_{i}^l = S_l e_i$, for $i \in \sigma_l$. Then

$$\langle z_{j}^l, z_{i}^l \rangle = \langle S_l e_j, S_l e_i \rangle = \langle B_l e_j, e_i \rangle = (b_{ij})^l$$

hence,

$$\left\| \sum_{i \in \sigma_l} c_i z_{i}^l \right\|^2 = \sum_{i, j \in \sigma_l} \langle c_i z_{j}^l, c_i z_{i}^l \rangle \quad (4.12)$$
as we computed before. \( (4.12-4.14) \) hold for every \( 1 \leq l \leq k \) and every choice of scalars \( \{c_i\}_{i \in \sigma_l} \). If we take in \( (4.12-4.14) \) \( c_i = 1 \) for a single index \( i \) and \( c_i = 0 \) for all the rest, we get:

\[
|x_l|^2 = \frac{M^2 - |x_l|^2}{M^2 - 1} \geq \frac{M^2 - 1}{M^2 - 1} = 1
\]  

(4.15)

because \( |x_l|^2 \leq 1 \) for all \( 1 \leq l \leq k \) and \( i \in \sigma_l \). Also, from (b) we have:

\[
\frac{1}{M^2 - 1} \left[ M \left( \sum_{i \in \sigma_l} |c_i|^2 \right)^{1/2} - \| \sum c_i x_l^t \| \right] \left[ M \left( \sum_{i \in \sigma_l} |c_i|^2 \right)^{1/2} + \| \sum c_i x_l^t \| \right] \leq \frac{1}{M^2 - 1} \left[ 2M \left( \sum_{i \in \sigma_l} |c_i|^2 \right)^{1/2} \right]^2
\]

\[
= \frac{4M^2}{M^2 - 1} \sum_{i \in \sigma_l} |c_i|^2
\]

Combining this with \( (4.12-4.14) \) gives:

\[
\| \sum_{i \in \sigma_l} c_i z_l^t \| \leq \frac{2M}{\sqrt{M^2 - 1}} \left( \sum_{i \in \sigma_l} |c_i|^2 \right)^{1/2} = \frac{2k^{1/2}}{(k-2)^{1/2}} \left( \sum_{i \in \sigma_l} |c_i|^2 \right)^{1/2}
\]  

(4.16)

(recall that \( M = \sqrt{k/2} \)). Consider the operators \( T_i : |1_{\sigma_l}| \rightarrow |1_{\sigma_l}| \) defined by \( T_i e_i = \frac{1}{\| \sigma_l \|} \). One has \( \| T_i e_i \| = 1 \) and by (4.15), (4.16) also \( \| T_i \| \leq \frac{2k^{1/2}}{(k-2)^{1/2}} \). At this point we use restricted invertibility. We get a constant \( c > 0 \) and partitions \( \eta_1, \ldots, \eta_m \) of each one of the sets \( \sigma_l, 1 \leq l \leq k \) such that for every choice of scalars \( c_i \):

\[
(1 \leq l \leq k, \ 1 \leq t \leq m) : \ \| \sum_{i \in \eta_t^l} c_i z_l^t \| \geq c \left( \sum_{i \in \eta_t^l} |c_i|^2 \right)^{1/2}
\]

From \( (4.12-4.14) \) we get:

\[
\frac{1}{M^2 - 1} \left( M^2 \sum_{i \in \eta_t^l} |c_i|^2 - \| \sum_{i \in \eta_t^l} c_i z_l^t \| \right) \geq c^2 \sum_{i \in \eta_t^l} |c_i|^2
\]
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hence

$$\left| \sum_{i \in \eta'_t} c_i x_i^j \right|^2 \leq M^2 \sum_{i \in \eta'_t} |c_i|^2 - (M^2 - 1)c^2 \sum_{i \in \eta'_t} |c_i|^2$$

$$= (M^2 - (M^2 - 1)c^2) \sum_{i \in \eta'_t} |c_i|^2$$

for all $1 \leq t \leq m$, $1 \leq l \leq k$. Denote $M_1 = (M^2 - (M^2 - 1)c^2)^{1/2}$. Then:

$$\left| \sum_{i \in \eta'_t} c_i x_i^j \right| \leq M_1 \left( \sum_{i \in \eta'_t} |c_i|^2 \right)^{1/2}$$

for all $1 \leq t \leq m$, $1 \leq l \leq k$, and every choice of scalars $c_i, i \in \eta'_t$. We now iterate the same process for the new partition. We will have obtained a further refinement of the partition, and a number $M_2$ given by:

$$M_2^2 = (1 + (M_1 - 1)(1 - c^2)) = 1 + (M^2 - 1)(1 - c^2)^2$$

Repeating this sufficiently times we shall get:

$$(M^2 - 1)(1 - c^2)^N < 1$$

and a refinement to a partition $\Lambda_1, \ldots, \Lambda_{m(\varepsilon)}$ so that

$$\left| \sum_{i \in \Lambda_j} c_i x_i^j \right| \leq (M^2 - 1)(1 - c^2)^N \left( \sum_{i \in \Lambda_j} |c_i|^2 \right)^{1/2}$$

and so

$$\left| \sum_{i \in \Lambda_j} \frac{1}{\|S\|} \sqrt{\frac{k}{2}} P_{\Lambda_j} S e_i \right| \leq (M^2 - 1)(1 - c^2)^N \left( \sum_{i \in \Lambda_j} |c_i|^2 \right)^{1/2}$$

therefore:

$$\| P_{\Lambda_j} S \sum_{i \in \Lambda_j} c_i e_i \| \leq \|S\| \left( \sqrt{\frac{2}{k}} \left( \sum_{i \in \Lambda_j} |c_i|^2 \right) \right)^{1/2}$$

and since we took $2\|S\|^2 / k < \varepsilon^2$, we get for all $x$:

$$\| P_{\Lambda_j} S P_{\Lambda_j} x \| \leq \varepsilon \|P_{\Lambda_j} x\| \leq \varepsilon \|x\|$$

so $\|P_{\Lambda_j} S P_{\Lambda_j}\| \leq \varepsilon$ for each $1 \leq j \leq m(\varepsilon)$. This completes the proof that restricted invertibility implies the local paving property, and of the equivalence theorem.
Bibliography


